Exam Stochastic Models (theory)
Monday May 18, 2020, 13.45 - 16.45 hours.

Modelling and analysis of stochastic processes for Math (201400434, coord. Scheinhardt)
Modelling and analysis of stochastic processes for IEM (201400062, coord. Mes)

Teachers: Boucherie, Timmer, Scheinhardt

This exam consists of **four exercises**. Please start each exercise on a new sheet of paper. Put your name and student number on **each sheet** you submit. **Motivate all answers** Use your time **efficiently**; e.g., correct miscalculations in time-consuming solutions only when you have time left; most points are awarded for showing your understanding.

Total points: 90. Grade = 1 + obtained points / 10

Integrity statement

Please read the following paragraph carefully, copy the text below it verbatim to the first page of your work (handwritten), add your full name and program (IEM/Math/...) and sign it. If you fail to do so, your test will not be graded.

By testing you remotely in this fashion, we express our trust that you will adhere to the ethical standard of behaviour expected of you. This means that we trust you to answer the questions and perform the assignments in this test to the best of your own ability, without seeking or accepting the help of any source that is not explicitly allowed by the conditions of this test.

The **only allowed sources** for this test are:

- the books by Winston and/or Ross as used in this module (hardcopy or pdf)
- the slides (printed or pdf)
- your own summaries/notes (but no solutions to tutorial/exam problems).
- an ordinary (scientific) calculator (not a programmable or graphic calculator)
- other electronic devices (laptop/tablet/mobile phone), but only to be used:
  - for downloading the test
  - to show the test/book/slides on your screen
  - to write the test (in case you prefer to use a tablet instead of paper to write on)
  - for making scans or photos of your work and uploading them to Canvas

Copy this text (handwritten), add your full name and program, and sign it:

I will make this test to the best of my own ability, without seeking or accepting the help of any source not explicitly allowed by the conditions of the test.

P.T.O.

1. **(25 points)** Customers arrive at a service center according to a Poisson process with rate $\lambda = 30$ per hour. The service center has 2 servers, server 1 and server 2. A customer requires service from one of these servers. The service time distribution of a customer at server 1 is exponential with mean $1/\mu_1 = 2$ minutes and at server 2 with mean $1/\mu_2 = 3$ minutes. If a customer arrives and both servers are free, then the customer selects the fast server. If the fast server is already serving a customer, then an arriving customer chooses to be served by the slower server. If there are 2 or more customers in the service center, then each server will be serving a customer.

   (a)–3 pt Consider a customer that arrives to the service center when the fast server is occupied and the slow server is free. Show that it is indeed preferable for a customer that arrives to the service center to be served by the slower server instead of waiting for the fast server to complete its service and then be served by the fast server.

   (b)–3pt Argue that on the long run a single server is not sufficient to serve the arriving customers.

   (c)–5pt We wish to study the evolution of the number of customers in the system. Let $X(t)$ be the number of customers present at time t. Explain whether the process $\{X(t),\ t > 0\}$ is a Markov chain or not. If so, give the transition diagram of $\{X(t),\ t > 0\}$. If not, then adapt the state description to arrive at a Markov chain, and give the transition diagram for that Markov chain. In either case, include a motivation for the transition rates in your diagram.

   (d)–3 pt Give the equations to determine the steady state distribution P of the Markov chain in part (c).

   (e)–3 pt Solve these equations to obtain the steady state distribution P.

   The answers to the next questions may be expressed in the arrival rate λ, and the service rates $\mu_1,\ \mu_2$ and the steady state distribution P. [The solution of (e) is therefore not required for these answers.]

   (f)–4 pt Give an expression for the utilization of the fast server. Give an expression for the utilization of the slow server. Motivate your answer.

   (g)–2 pt Give an expression for the average number of waiting customers (not including those that are being served). Motivate your answer.

   (h)–2 pt Give an expression for the fraction of customers who do not need to wait before their service starts. Motivate your answer.

2. **(20 points)** Consider an open network of three stations. Every station contains a single server, and all arriving customers arriving at a station may enter the station. Service is in order of arrival. The service times at stations 1, 2 and 3 have exponential distributions with rates $\mu_1 = 1$, $\mu_2 = 2$ and $\mu_3 = 3$, respectively. The external arrival rates of customers to stations 1, 2 and 3 are $\gamma_1 = \gamma, \gamma_2 = 0$ and $\gamma_3 = \gamma$, respectively. Customers that complete service at station i route to station j with probabilities r_{ij}, $i, j = 1, 2, 3$, that are determined as follows

$$r_{12} = \frac{1}{2},\ r_{13} = 0,\ r_{21} = \frac{1}{3},\ r_{23} = 0,\ r_{31} = 0,\ r_{32} = \frac{1}{4}.$$

(a)–6 pt Give the traffic equations and solve the traffic equations.

(b)–3 pt Give the stability condition for the network. Motivate your answer.

(c)–3 pt Give the joint distribution of the number of customers at the three stations. Motivate your answer.

(d)–4 pt Now consider the part of the network consisting of stations 1 and 2. Give an expression for the mean sojourn time of a customer in this part of the network. Motivate your answer.

(e)–4 pt Now assume that the distribution of the service time in station 3 is changed into the following (Hyperexponential) distribution: with probability 1/4 the service time has an exponential distribution with rate 3/2 and with probability 3/4 an exponential distribution with rate 9/2. What is the mean sojourn time of a customer in station 3? Motivate your answer.

3. **(24 points)** To earn some money, Donald decides to take part in a game. In this game, he has to spin a wheel, which has numbers 1 to 4 marked on it. The probability that the wheel stops at number j is p_j, see the table below.

j	1	2	3	4
p_j	0.34	0.28	0.22	0.16

He has to pay 5 euros to spin the wheel *up to* 5 times. The payoff is two times the number produced in the *last* spin in euros. Donald wants to maximise his profit resulting from this game.

(a)–6 pt This problem can be formulated as a stochastic dynamic program. What do you choose as stages, states, decisions and optimal value function?

(b)–4 pt What is the recurrence relation of the optimal value function?

(c)–3 pt How many policies are available to Donald? And how many of these are stationary policies?

(d)–9 pt Use dynamic programming to solve the problem. What is Donald's largest expected profit? Which policy results in this value?

(e)–2 pt Suppose now that Donald changes his goal: he wants to maximise the probability of achieving an expected net profit of at least 1 euro. What should change to your formulation of the stochastic dynamic program in (a)? (You need not solve it.)

4. **(21 points)** Webshop BuyHere seeks to boost its online sales. The shop knows from experience that advertisements on social media have a positive effect on its revenue in the short run. In particular, it is known that the revenue in week $n+1$ depends on the revenue in week n as well as any advertisements in week $n+1$ (see the following table).

	revenue week $n+1$					
	advertisements			no advertisements		
revenue week n	high	middle	low	high	middle	low
high	0.8	0.2	0	0.5	0.5	0
middle	0.7	0.3	0	0	0.8	0.2
low	0.5	0.5	0	0	0.2	0.8

For instance, when the current revenue in week n is high and BuyHere decides not to place advertisements in week $n+1$, the probability of high revenue in week $n+1$ is 0.5. Weekly advertisements cost 6000 euros. The expected revenue at high, middle and low revenue are 15000, 7500 and 2000 euros, respectively. The webshop aims to maximise its expected discounted profit over an infinite horizon at discount factor 0.9 per week.

(a)–4 pt Model this problem as a Markov decision problem. What do you choose as states, decisions and optimal value function?

(b)–4 pt Determine the expected direct rewards for each state and decision.

(c)–4 pt Formulate the linear programming model that BuyHere may use to determine the optimal profit.

(d)–2 pt Currently, BuyHere uses the following stationary policy. They advertise on social media only if last week's revenue was low; otherwise they do not advertise. Formulate the equations that you have to solve to determine the value of this policy. (You need not solve these.)

(e)–4 pt The values that solve the equations in (d) are respectively 75945, 67822 and 69945 euros for the states high, middle and low, respectively (rounded to integer numbers). Use policy iteration to determine if the policy given in (d) is optimal.

(f)–3 pt Now assume that BuyHere follows a very simple policy in which advertisements are placed every week, regardless of the revenues that are obtained. Starting with a low revenue in week 0, what is the expected number of weeks until the revenue is high for the first time?