Exam Limits to Computing (201300042)

Thursday, November 3, 2022, 8:45 – 11:45

- You can bring five sheets of A4 paper with arbitrary content to the exam.
- Electronic devices of any kind, books, ... are not allowed.
- This exam consists of four problems.
- Please start a new page for each problem.
- The total number of points is 60. Sufficient for passing are 30 points.

1. Computability

Let

$$\mathrm{FP} = \{g \in \mathcal{G} \mid \varphi_g(g) = g\}.$$

(a) *(8 points)* Is $\mathrm{FP} \in \mathsf{RE}$? Prove your answer.

(b) *(8 points)* Is $\mathrm{FP} \in \mathsf{REC}$? Prove your answer.

(c) *(2 points)* Is $\mathrm{FP} \in \mathsf{co\text{-}RE}$? Prove your answer.

(d) *(5 points)* Prove that FP is not an index set.

2. NP-Completeness

An instance of SetCover is a finite set U together with subsets $S_1, \ldots, S_m \subseteq U$ and a $k \in \mathbb{N}$. The question is if there exists k of these subsets that cover all elements in U.

More formally, an instance as described above is a "yes" instance if there exists a set $I \subseteq \{1, \ldots, m\}$ with $|I| \leq k$ such that

$$\bigcup_{i \in I} S_i = U.$$

(13 points) Prove that SetCover is NP-complete.

Hint: VertexCover $= \{(G, k) \mid$ undirected graph G has a vertex cover of size $k\}$ is NP-complete.

3. Complexity

Let $\mathsf{E} = \mathsf{DTime}(2^{O(n)})$. Recall that $\mathsf{EXP} = \bigcup_{c>0} \mathsf{DTime}(2^{O(n^c)})$.

(a) *(2 points)* Prove that $\mathsf{E} \subsetneq \mathsf{EXP}$.

(b) *(4 points)* Prove that E is not closed under polynomial-time many-one reductions. This means that there are problems $A \notin \mathsf{E}$ and $B \in \mathsf{E}$ with $A \leq_{\mathrm{P}} B$.

Hint: Consider some $A \in \mathsf{EXP} \setminus \mathsf{E}$ and $B = \{x\#1^{f(|x|)} \mid x \in A\}$ for some well-chosen function f.

(c) *(2 points)* Prove that $\mathsf{E} \neq \mathsf{PSPACE}$.

Hint: You cannot conclude $\mathsf{E} \subsetneq \mathsf{PSPACE}$ or $\mathsf{PSPACE} \subsetneq \mathsf{E}$ from this. You can only conclude that the two classes are not equal.

4. Questions

Are the following statements true or false? Justify your answers briefly.

(a) *(2 points)* $\mathsf{co\text{-}NSpace}(n^2) \subsetneq \mathsf{DSpace}(n^5)$.

(b) *(2 points)* For all $L \subseteq \mathbb{N}$, the following holds: If $L \in \mathsf{REC}$, then χ_L is total.

(c) *(2 points)* For all $L \subseteq \mathbb{N}$, the following holds: If χ_L is total, then $L \in \mathsf{REC}$.

(d) *(2 points)* If $L \leq H_0$, then L is recursively enumerable.

(e) *(2 points)* If $\mathsf{P} = \mathsf{NP}$, then 2SAT is NP-complete.

(f) *(2 points)* If 2SAT is NP-complete, then $\mathsf{P} = \mathsf{NP}$.

For the last two parts of this exercise, let $\mathsf{QP} = \mathsf{NP} \cup \mathsf{co\text{-}NP}$.

(g) *(2 points)* QP is closed under polynomial-time many-one reductions.

(h) *(2 points)* If there exists a QP-complete problem, then $\mathsf{NP} = \mathsf{co\text{-}NP}$.