Module 2, resit Analysis, 201300057

1-2-2017

13:45h - 16:45h

Motivate all your answers.

1. Decide which of the following statements are true and which are false. Prove the true ones and provide counterexamples for the false ones.

   (a) (3pt) If a is an upper bound of a set $E \subseteq \mathbb{R}$ and $a \in E$, then a is the supremum of E.

   (b) (3pt) Let X be a set and $\{E_\alpha\}_{\alpha \in A}$ be a collection of subsets of X, then

   $$\left(\bigcup_{\alpha \in A} E_\alpha\right)^c = \bigcap_{\alpha \in A} E_\alpha^c$$

   (c) (3pt) Let $\{x_n\}$ and $\{y_n\}$ be sequences of real numbers. If x_n converges to zero and $y_n > 0$ for all $n \in \mathbb{N}$, then $x_n y_n$ converges.

2. (a) (3pt) Use the sequential characterization of limits to prove that

   $$f(x) = \begin{cases} \sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

   has no limit as $x \to 0$.

   (b) (4pt) Let f be a function that is continuous in point a and $f(a) \geq 0$. Prove that there exist a $\delta > 0$ such that $f(x) > 0$ if $|x - a| < \delta$.

3. (5pt) Formulate Rolle's Theorem and prove this theorem.

4. (a) (2pt) Formulate Taylor's Theorem.

   (b) (2pt) Let $f(x) = \log(x)$ and $n \in \mathbb{N}$. Find the Taylor polynomial $P_n := P_n^{f,1}$.

   (c) (3pt) Prove that if $x \in [1, 2]$, then

   $$|\log(x) - P_n(x)| \leq \frac{1}{n+1}$$

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5. (a) (3pt) If f is increasing on $[a, b]$ and $P = \{x_0, x_1, \cdots, x_n\}$ is any partition of $[a, b]$, prove that

$$\sum_{j=1}^{n}(M_j(f) - m_j(f))\Delta x_j \leq (f(b) - f(a))\|P\|.$$

(b) (3pt)

Prove that if f is increasing on $[a, b]$, then f is integrable on $[a, b]$.

(c) (2pt)

Prove that if f is monotone on $[a, b]$, then f is integrable on $[a, b]$.

Total: 36 points

$b \geq x_n \geq x_0 \geq a$

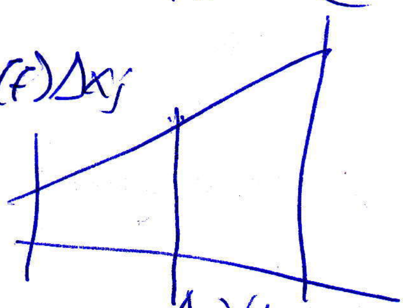
$x_n = b \quad x_0 = a$

$x_n - x_0 \leq f(b) - f(a) \qquad \|P\| \geq \Delta x_j$

$U(f,P) - L(f,P) < \varepsilon \qquad \|P\| < \delta$

$\delta = \dfrac{\varepsilon}{f(b) - f(a)}$

$U(f,P) = \sum_{j=1}^{n} M_j(f)\Delta x_j$



Δx_j

$M_r(f) - m_r(f)$