Module 2, Exam Analysis, 201700140

19-01-2018

13:45h - 16:45h

Motivate all your answers.

1. Decide which of the following statements are true and which are false. Prove the true ones and provide counterexamples for the false ones.

   (a) (3pt) Suppose A and B are nonempty subsets of $\mathbb{R}$ and $A \subseteq B$. If B has a supremum, then $\sup A \leq \sup B$.

   (b) (3pt) If $\{x_n\}$ is Cauchy and $\{y_n\}$ bounded, then $\{x_n y_n\}$ is Cauchy.

   (c) (3pt) Suppose that $f : [a, b] \to \mathbb{R}$ and $c \in (a, b)$. If f is differentiable on $[a, c]$ and f is differentiable on $[c, b]$, then f is differentiable on $[a, b]$.

2. (a) (3pt) Let $\{x_n\}$ be a sequence in $\mathbb{R}$.
   Prove that if $\lim_{n\to\infty} x_n = a$, then $\lim_{n\to\infty} |x_n| = |a|$.

   (b) (3pt) Prove the following statement:
   Any convergent sequence is bounded.

3. (a) (2pt) Let E be a nonempty subset of $\mathbb{R}$. Give the definition of $f : E \to \mathbb{R}$ *is uniformly continuous on* E.

   (b) (2pt) Prove, using the definition of part (a), that $f(x) = x^2 + x$ is uniformly continuous on $(0, 1)$,

   (c) (2pt) Prove that $f(x) = \dfrac{\sin(x)}{x}$ is uniformly continuous on $(0, 1)$.

4. (a) (2pt) Formulate the Mean Value Theorem.

   (b) (4pt) Suppose f is a function on $[0, \infty)$ and f has the following properties:

   - $f(0) = 0$;
   - f is continuous on $[0, \infty)$;
   - f' exists for $x > 0$;
   - f' is increasing.

   Define $g(x) = \dfrac{f(x)}{x}$ for $x > 0$.

   Prove that for $x > 0$ we have $f'(x) \geq \dfrac{f(x)}{x}$ and use this to prove that g is increasing.

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5. Let $f : [0,1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & \text{if } x = \dfrac{1}{n} \text{ with } n \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}.$$

(a) (3pt) Show that for any partition P of $[0,1]$: $L(f,P) = 0$.

(b) (3pt) Show that for each $\epsilon > 0$ there exists a partition P such that $U(f,P) < \epsilon$.

(c) (3pt) Prove that f is Riemann-integrable on $[0,1]$ and determine the integral.

Total: 36 points