Module 2, Exam Analysis, 201700140

31-01-2018

13:45h - 16:45h

Motivate all your answers.

1. Decide which of the following statements are true and which are false. Prove the true ones and provide counterexamples for the false ones.

   (a) (3pt) Let X be a set and $\{E_\alpha\}_{\alpha \in A}$ be a collection of subsets of X, then
   $$\left(\bigcap_{\alpha \in A} E_\alpha \right)^c = \bigcup_{\alpha \in A} E_\alpha^c$$

   (b) (3pt) Suppose $\{x_n\}$ is a bounded sequence.
   Then, for all $k \in \mathbb{N}, k \geq 1$: $\dfrac{x_n}{n^k} \to 0$ as $n \to \infty$.

   (c) (3pt) Let $a \in \mathbb{R}$ and let f and g be real functions defined at all points x in some open interval I containing a except possibly at $x = a$.
   If $\lim_{x \to a} f(x)$ does not exist and $f(x) \leq g(x)$ for all $x \in I$, then $\lim_{x \to a} g(x)$ doesn't exist either.

2. (a) (2pt) Let $E \subseteq \mathbb{R}$ be nonempty.
   Give the definition of *supremum* of E.

   (b) (3pt) Prove the Approximation Property for Suprema:
   If E has a finite supremum and $\epsilon > 0$ is any positive number, then there is a point $a \in E$ such that
   $$\sup(E) - \epsilon < a \leq \sup(E).$$

3. (a) (2pt) Formulate the Intermediate Value Theorem.

   (b) (4pt) Let $f : (1, 2] \to \mathbb{R}$ be defined by
   $$f(x) = \frac{1}{x^2 - 1}.$$
   Use the definition of limits to prove that $\lim_{x \to 1+} f(x) = \infty$.

   (c) (3pt) Prove that there is a $c \in (1, 2)$ such that $f(c) = c$.

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4. (a) (3pt) Suppose $f \in \mathcal{C}^{\infty}(a,b)$ and $x_0 \in (a,b)$. Give the Taylor polynomial $P_n^{f,x_0}(x)$ of order n by f centered at x_0. Give an upperbound for the error $|f(x) - P_n^{f,x_0}(x)|$?

   (b) (3pt) Suppose f is defined at an interval I that contains a and suppose $f \in \mathcal{C}^2(I)$. Show that

$$\lim_{h \to 0} \frac{f(a+h) + f(a-h) - 2f(a)}{h^2} = f''(a)$$

   (*hint: use L'Hospital's rule.*)

5. For each $n \in \mathbb{N}$ define $P_n = \{\frac{j}{n} \mid j = 0, 1, 2, \cdots, n\}$.

   Let $f : [0,1] \to \mathbb{R}$ be given by $f(x) = x^2$

   (a) (1pt) Show that for each $n \in \mathbb{N}$: P_n is a partition of $[0,1]$.

   (b) (4pt) Give a formula for $U(f, P_n)$ and $L(f, P_n)$ and use these to show that f is integrable on $[0,1]$ and compute the value of $\int_0^1 f(x)dx$.

   (*Hint: use the following sum:* $\sum_{k=1}^{n} k^2 = \dfrac{n(n+1)(2n+1)}{6\cancel{n^3}}$.)

   (c) (2pt) Is $g : [0,1] \to \mathbb{R}$ with $g(x) = x^2 \sin(\frac{1}{x})$ as $x \neq 0$ and $g(0) = 0$ integrable on $[0,1]$?
   If so, give an upperbound M of the value of the integral $\int_0^1 g(x)dx$, with $0 < M < 1$. Motivate your answer.

Total: 36 points