Exam: Analysis-I (202001329) MOD-02-AM: Mathematical Proof Techniques

Instructor: Pranab Mandal 14-January-2021, 09:00 – 12:00 Total Points : 35

All answers must be motivated.
Approach to a solution is equally important as the final answer.
Use of an electronic calculator or a book is not allowed.
Good Luck!

1. Suppose V and W are two nonempty subsets of $\mathbb{R}$, which are bounded above. We define the sum of V and W as $V + W = \{v + w \mid v \in V,\ w \in W\}$.

   (a) Show that $V + W$ has a supremum. [2]

   (b) Show that $\sup(V + W) = \sup(V) + \sup(W)$. [5]

2. Let the real sequence $\{a_n\}_{n\in\mathbb{N}}$ be defined as follows: $a_1 = 3$ and

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{7}{a_n}\right), \quad n \geq 1.$$

   (a) Show that $\sqrt{7} < a_{n+1} < a_n$ for all $n \in \mathbb{N}$. [5]

   (b) Show that the sequence $\{a_n\}_{n\in\mathbb{N}}$ converges and find its limit. [1+2]

3. (a) Suppose f is continuous on $[a, b]$ and $x_n \in [a, b]$, $(n \in \mathbb{N})$ is a convergent sequence. Prove that $f(x_n)$ converges as $n \to \infty$. [4]
   [You may not use the sequential characterization of limits.]

   (b) Suppose f is defined on $\mathbb{R}$ and is given by: $f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$

   Prove that $f(x)$ has no limit as $x \to 0$. [4]

4. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function satisfying $f(0) = 0$ and $f'(0) = 1$. Show that f takes both positive and negative values. [4]

5. (a) Suppose $f[a, b] \to \mathbb{R}$ is nonnegative and continuous on $[a.b]$.
   Show that $\int_a^b f(x)\,dx = 0$ if and ony if $f(x) = 0,\ \forall x \in [a, b]$. [5]

   (b) Is the statement true if the nonegativity condition on f is removed? Justify. [1]

   (c) Prove, however, that the following holds. If f is any integrable function on $[a.b]$, then $\int_a^x f(u)\,du = 0,\ \forall x \in [a, b]$ if and ony if $f(x) = 0,\ \forall x \in [a, b]$. [2]

Grade:	$\dfrac{\text{obtained score}}{35} \times 9 + 1$ (rounded off to one decimal place)