Exam: Analysis-I (202001329)

Instructor: Pranab Mandal   MOD-02-AM: Mathematical Proof Techniques

04-February-2022, 13:45 - 16:45

Total Points : 40

- Closed book exam! Use of own text-material or an electronic calculator is not allowed.
- All answers must be motivated. For the short-answer questions (1-2), a brief one is enough.
- For the other questions (3-5), you *must* follow the four steps (practised during TBL).
  (i.) Get Started: describe what the problem is and your initial thoughts about it
  (ii.) Devise Plan: provide an outline how you are going to solve (or have solved) the problem
  (iii.) Execute: execute your plan (and try) to reach your solution
  (iv.) Evaluate: reflect on your solution and/or approach

  Points are distributed (roughly) as: steps (i.)+(ii.) 40%, step (iii.) 40% and step (iv.) 20%. Exact allocations are mentioned next to each (part of a) question.
- Good Luck!

[Short-answer questions]

1. Suppose X, Y are some sets and $f : X \to Y$ is a function. Recall the notations $f(A)$ and $f^{-1}(B)$ for the forward and inverse images, respectively, under f where $A \subseteq X$ and $B \subseteq Y$. Show that the following statement is not true. [4]

$$f^{-1}(f(E)) = E, \quad \text{for } E \subseteq X.$$

2. Prove, from the definition, that $\lim_{n\to\infty} a_n = 0$, where $a_n = \dfrac{(-1)^n + \sin(n^3)}{n^2}$, for $n \in \mathbb{N}$. [4]

[Four-step questions]

3. Find $\lim_{x\to\infty} \left(\sqrt[3]{x^2(x+3)} - x \right)$ through the following steps. [5+5+2]

   (a) Give the complete statement of the Mean Value Theorem (for derivative).

   (b) Prove and use the following relation to find the desired limit.

$$\frac{1}{\sqrt[3]{(x+3)^2}} \le \sqrt[3]{x+3} - \sqrt[3]{x} \le \frac{1}{\sqrt[3]{x^2}}, \quad \text{for } x > 0.$$

4. Let $[a, b]$ be a non-degenerate interval and $f : \mathbb{R} \to \mathbb{R}$ is continuous and one-to-one on $[a, b]$. Prove that if $f(a) < f(b)$, then f is increasing on $[a, b]$. [4+4+2]

5. Prove that the function $f : [0, 1] \to \mathbb{R}$, given as below, is integrable on $[0, 1]$. [4+5+1]

$$f(x) = \begin{cases} e^{-n} & \text{if } x = 2^{-n} \text{ for some } n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Grade:	$\dfrac{\text{obtained score}}{40} \times 9 + 1$