Resit Linear Optimization (course code 201300057)

Give motivations for all your answers.

1. a) You want to make as much money as possible per day (16 hours, since you have to sleep for 8 hours). To make money, you can do hard tasks (earning €20 per hour) and easy tasks (earning €5 per hour). 1 hour of hard tasks requires 1000 kJ of energy, while 1 hour of easy tasks requires 500 kJ. You have 5000 kJ per day, but can can obtain more energy by cooking food (Cooking for 1 hour yields 1000 kJ of energy, but costs €5), and by ordering fast food (which costs no time, but 800 kJ of energy costs €15). Assume that you can eat while doing tasks. Formulate a linear program that finds the largest profit (earnings minus losses) you can make per day.

   b) Suppose someone offers you coffee. Each liter of coffee would allow you to sleep one hour less. Explain how you can use the dual of the linear program to find how much money coffee is worth to you. (You do not have to compute the dual or solve it. Simply state the step(s) you would take and the condition(s) under which you can use the dual to find the value of coffee.)

2. a) Solve by the Simplex Method:

$$
\begin{array}{llllll}
\min & -x_1 & -2x_2 & +x_3 & \\
\text{s.t.} & & x_2 & -x_3 & \leq 1 \\
 & 2x_1 & +x_2 & +x_3 & \leq 5 \\
 & x_1 & & +3x_3 & \leq 10 \\
 & & \mathbf{x} \geq 0 & &
\end{array}
$$

   b) Use the final tableau to find a different optimal solution, or to argue that no such solution exists.

Turn the page for exercises 3 and 4!

3. For each of the following statements, determine whether it is true or false. Provide either a short argument of at most two lines, or a (counter)example.

   a) Consider a non-empty n-dimensional polyhedron P. For every $\mathbf{c} \in \mathbb{R}^n$, there exists a vertex $\mathbf{x}^* \in P$ such that $\mathbf{x}^*$ maximizes $\mathbf{c}'\mathbf{x}$ over all $\mathbf{x} \in P$.

   b) At any degenerate basic feasible solution of a linear program in standard form, there exists a basic direction that is not feasible.

   c) Suppose that during some step of the simplex method, we move to a non-degenerate vertex. Then in this step, any pivoting rule (including for example the lexicographic pivoting rule) would choose the same variable to exit the basis.

   d) In the first phase of the 2-phase simplex method, giving y_1 a cost of 2 also works, i.e., the following linear program will either yield a feasible solution to the original problem $\{\min \mathbf{c}'\mathbf{x}, \text{s.t.} A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0\}$, or determine that no such solution exists:

$$\begin{aligned} \min \quad & 2y_1 + y_2 + y_3 + \cdots + y_n \\ \text{s.t.} \quad & A\mathbf{x} + \mathbf{y} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0}, \mathbf{y} \geq \mathbf{0} \end{aligned}$$

4. Consider the following linear program:

$$\begin{aligned} \min \quad & x_1 + 2x_2 \\ \text{s.t.} \quad & x_1 + x_2 \geq 3 \\ & \mathbf{x} \geq 0 \end{aligned}$$

Show that its dual is equivalent to the dual we obtain by considering the non-negativity constraints as regular constraints, i.e., the dual of the following linear program:

$$\begin{aligned} \min \quad & x_1 + 2x_2 \\ \text{s.t.} \quad & x_1 + x_2 \geq 3 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \\ & \mathbf{x} \text{ free} \end{aligned}$$

exercise	1	2	3	4
points	9	10	8	5