Test Linear Optimization

Monday, January 22, 2018, 8.45-11.45

Give motivations for all your answers. You are not allowed to use a calculator.

1. Consider the optimal currency conversion problem: Suppose there is a set $N = \{1, \dots, n\}$ of available currencies, and assume that one unit of currency i can be exchanged for r_{ij} units of currency j. (Naturally, we assume that $r_{ij} > 0$.) There are certain regulations that impose a limit u_i on the total amount of currency i that we can exchange into other currencies on any given day. Suppose that we start with B units of currency 1 and that we would like to maximize the number of units of currency n that we end up with at the end of the day, through a sequence of currency transactions. Assume that for any sequence $i_1, \dots, i_k$ of currencies, we have $r_{i_1 i_2} \cdot r_{i_2 i_3} \cdot \dots \cdot r_{i_{k-1} i_k} \cdot r_{i_k i_1} \leq 1$, which means wealth cannot be increased by going through a cycle of currencies.

   Let $T = \{1, \dots r\}$ denote a set of points in time.
   Let c_{it} denote the amount of currency i we have at time $t \in T$.
   Let e_{ijt} denote the amount of currency i that we exchange to currency j between time points t and $t+1$.
   We want to use the following linear program to solve the problem:

$$
\begin{array}{lll}
\max c_{nr} & & \\
\text{s.t.} & c_{j,t+1} = \sum_{i \in N \setminus \{j\}} e_{ijt} \cdot r_{ij} & \forall j \in N, \forall t \in T \\
& \sum_{j \in N \setminus \{i\}} e_{ijt} \leq c_{it} & \forall i \in N, \forall t \in T \\
& \sum_{j \in N \setminus \{i\}} \sum_{t \in T} e_{ijt} \leq u_i & \forall i \in N, \forall t \in T \\
& c_{11} = B &
\end{array}
$$

   The first set of constraints states that the amount of currency j we have at each time point equals the amount of currency we converted to j since the previous time point. The second set of constraints states that the amount of currency i we can convert after each time point equals at most the amount of currency i we had at that time point.
   The third set of constraints expresses the regulations that impose a limit u_i.
   The fourth (single) constraint states that we start with B units of currency 1.

   a) Find all the mistakes in this linear program, explain why they are wrong, and explain how you would correct them.

   b) Explain intuitively why we can choose $r = n$, i.e. we do not need more than n time points.

2. **Prove** that the function $f(x) = |x|, x \in \mathbb{R}$ is convex.

3. Suppose that $\mathbf{x}$ and $\mathbf{y}$ are vertices of some polyhedron P. Prove that $2\mathbf{y} - \mathbf{x}$ is not a vertex of P.

   Turn the page for exercises 4, 5 and 6!

4. Solve the following linear program using the simplex method.

$$\begin{array}{ll} \max & 4x_1 - 2x_2 \\ \text{s.t.} & x_1 - x_2 \leq 1 \\ & 4x_1 - 4x_2 \leq 2 \\ & -4x_1 + 7x_2 \leq 1 \\ & x_1 \geq 0, x_2 \geq 0 \end{array}$$

5. Consider the following tableau:

$$\begin{array}{c|cccc} 3 & b & c & 0 & 0 \\ \hline 4 & a & 2 & 1 & 0 \\ 2 & 4 & 1 & 0 & 1 \end{array}$$

a) Suppose $b < 0$ and $c > 0$. For what values of a would the lexicographic pivoting rule pick the third column to leave the basis in the next iteration?

b) Suppose $b > 0$ and $c < 0$. For what values of a would the lexicographic pivoting rule pick the third column to leave the basis in the next iteration?

c) Do there exist values for a, b, c such that this is a tableau in the first step of the two-phase simplex method? If there do exist such values, give these values, and explain why. If not, explain why.

6. Consider the following linear program with optimum $x_1 = 3, x_2 = -1, x_3 = -3$.

$$\begin{array}{lrrrl} \max & 10x_1 & -5x_2 & +6x_3 & \\ \text{s.t.} & x_1 & -2x_2 & +x_3 & \leq 2 \\ & 2x_1 & -x_2 & +x_3 & \leq 4 \\ & 3x_1 & +3x_2 & +x_3 & \leq 3 \\ & -x_1 & -2x_2 & -x_3 & \leq 4 \\ \multicolumn{5}{l}{x_1 \geq 0, x_2 \leq 0, x_3 \leq 0} \end{array}$$

a) Construct the dual program.

b) Use complementary slackness to compute the optimum of the dual program.

exercise	1	2	3	4	5	6
points	8	8	8	8	8	8