Resit Linear Optimization

Friday, February 1, 2019, 13.45-16.45

Give motivations for all your answers. You are not allowed to use a calculator.

1. We can produce two different products a and b, using two resources, r and s.
   One unit of product a requires r_a units of resource r and s_a units of resource s and yields p_a dollars.
   One unit of product b requires r_b units of resource r and s_b units of resource ~~b~~ s and yields p_b dollars.
   We buy our resources from a supplier, who charges us c_r dollars per unit of resource r and c_s dollars per unit of resource s.
   Due to demand, we can sell at most d_a units of product a and d_b units of product b.

   8 a) Construct a linear program that maximizes our profit (yield minus cost).

   8 b) Suppose that our supplier offers us 10% discount if we spend at least t dollars on resources. Note that we can choose whether or not to make use of this offer. Explain how you would find the maximum profit in this case.
   Hint: Construct a new linear program, and use it together with the linear program from part a).

2. 8 a) Prove that the set $\{(x, y) \in \mathbb{R}^2 \text{ for which } -|x| \leq y\}$ is not convex.

   8 b) Prove that the set $\{(x, y) \in \mathbb{R}^2 \text{ for which } y \leq -|x|\}$ is convex.

b is a scalar, not a vector

3. 8 a) Prove or give a counterexample:
   For all polyhedra $P \subseteq \mathbb{R}^2$, for all vertices $\mathbf{x}^* \in P$, there exists a line $l = \{\mathbf{x} \in \mathbb{R}^2 | \mathbf{a}'\mathbf{x} = \mathbf{b}\}$ for which $l \cap P = \mathbf{x}^*$, i.e. $\mathbf{x}^*$ is the only element of P on l.

   8 b) Prove or give a counterexample:
   For all polyhedra $P \subseteq \mathbb{R}^2$, for all feasible solutions $\mathbf{x}^* \in P$, the following holds: If there exists a line $l = \{\mathbf{x} \in \mathbb{R}^2 | \mathbf{a}'\mathbf{x} = \mathbf{b}\}$ for which $l \cap P = \mathbf{x}^*$, then $\mathbf{x}^*$ is a vertex of P.

4. 8 Solve the following linear program using the simplex method.

$$\begin{array}{llll} \min & -4x_1 & +2x_2 & \\ \text{s.t.} & 2x_1 & -2x_2 & \leq 2 \\ & 4x_1 & -3x_2 & \leq 5 \\ & -x_1 & +2x_2 & \leq 3 \\ & \multicolumn{3}{l}{x_1 \geq 0, x_2 \geq 0} \end{array}$$

Turn the page for exercises 5 and 6!

5. Consider the following linear program in standard form (Phase 2):

$$\begin{array}{ll} \min & \mathbf{c}'\mathbf{x} \\ \text{s.t.} & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{array}$$

Here matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$ has m linearly independent rows, and vector $\mathbf{b} \in \mathbb{R}^m$, $\mathbf{b} \geq \mathbf{0}$. Recall that in the first phase of the two-phase simplex method, we introduce a vector $\mathbf{y} = (y_1, \ldots, y_m)$, and solve the following linear program (Phase 1):

$$\begin{array}{ll} \min & \sum_{i=1}^m y_i \\ \text{s.t.} & \mathbf{A}\mathbf{x} + \mathbf{y} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \\ & \mathbf{y} \geq \mathbf{0} \end{array}$$

8 a) Explain how we can find a basic feasible solution to the Phase 1 problem.

8 b) Explain for which results of the Phase 1 problem, we can conclude that the Phase 2 problem is infeasible.

8 c) Explain for which results of the Phase 1 problem, we can conclude that the Phase 2 problem has a basic feasible solution, and explain how to find this basic feasible solution.

8 d) Instead of $\min \sum_{i=1}^m y_i$, we can also use a different objective in Phase 1. For both of the following two objectives, explain intuitively whether you can conclude from an optimal solution of Phase 1, whether the Phase 2 problem has a feasible solution or not.
Objective 1: $\min \sum_{i=1}^m i \cdot y_i$. Objective 2: $\min \sum_{i=1}^m -y_i$

6. Consider the following linear program with optimal solution $x_1 = 30/17, x_2 = -11/17, x_3 = 0$.

$$\begin{array}{llllll} \min & 7x_1 & -x_2 & +2x_3 & \\ \text{s.t.} & x_1 & -5x_2 & +x_3 & \geq 5 \\ & 3x_1 & +2x_2 & +x_3 & \geq 4 \\ & 2x_1 & +2x_2 & -x_3 & \geq 1 \\ & x_1 & +3x_2 & & \geq -1 \end{array}$$

$x_1 \geq 0, x_2 \leq 0, x_3 \leq 0$

8 a) Construct the dual program.

8 b) Use complementary slackness to compute an optimal solution of the dual program.

exercise	1	2	3	4	5	6
points	8	8	8	8	8	8