Test Linear Optimization

Monday, January 21, 2019, 8.45-11.45

Give motivations for all your answers. You are not allowed to use a calculator.

1. We have vectors $\mathbf{v}_1, \ldots \mathbf{v}_n \in \mathbb{R}^m$ with costs $c_1, \ldots, c_n$ and a vector $\mathbf{b} \in \mathbb{R}^m$. Our goal is to find the cheapest linear combination that equals $\mathbf{b}$. For example, if $2\mathbf{v}_1 - 0.5\mathbf{v}_2 = \mathbf{b}$, this linear combination has a cost of $2c_1 - 0.5c_2$.

   8 a) Construct a linear program that finds the cheapest linear combination that equals $\mathbf{b}$.

   b) Suppose that we also require that all the vectors with non-zero coefficients are linearly independent. How can we adjust and/or use this linear program to find the cheapest linearly independent linear combination that equals $\mathbf{b}$?

4 5 2. **Prove** that the set $\{(x,y) \in \mathbb{R}^2 | x^2 + y^2 \leq 1\}$ is convex. You may use that $(a^2 + b^2 \leq 1$ and $c^2 + d^2 \leq 1)$ implies that $ac + bd \leq 1$ without proving it.

3 3. Consider the following linear program in standard form, that is equivalent to an LP where all variables are free:

$$\begin{array}{ll} \max & \mathbf{c}'\mathbf{x}_+ - \mathbf{c}'\mathbf{x}_- \\ \text{s.t.} & \mathbf{A}\mathbf{x}_+ - \mathbf{A}\mathbf{x}_- \leq \mathbf{b} \\ & \mathbf{x}_+ \geq \mathbf{0}, \mathbf{x}_- \geq \mathbf{0} \end{array}$$

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   Prove that if this LP has a finite optimum, then the feasible region contains an optimal solution that is not a vertex.

8 4. Solve the following linear program using the simplex method.

$$\begin{array}{lrrrl} \min & -3x_1 & +x_2 & +x_3 & \\ \text{s.t.} & 3x_1 & -3x_2 & +x_3 & \leq 3 \\ & x_1 & +2x_2 & & \leq 4 \\ & & 2x_2 & +x_3 & \leq 1 \\ & \multicolumn{4}{l}{x_1 \geq 0, x_2 \geq 0, x_3 \geq 0} \end{array}$$

8

Turn the page for exercises 5 and 6!

5. Consider the following tableau:

a	b	-2	0	0
2	4	c	1	0
1	-3	1	0	1

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a) Do there exist values for a, b, c such that this tableau is an initial tableau of the first phase of the two-phase simplex method? If there do exist such values, give all such values, and explain why. If not, explain why.

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b) Suppose we use the following pivoting rules: If multiple columns are candidates for entering the basis, we select the column with the lowest index among them to enter the basis. If multiple columns are candidates for leaving the basis, we use the lexicographic pivoting rule to select a column to leave the basis. Do there exist values for a, b, c such that column 3 leaves the basis? If there do exist such values, give all such values, and explain why. If not, explain why.

6. Consider the following linear program:

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$$
\begin{aligned}
\max\ & x \\
\text{s.t.}\quad & x = 2 \\
& x \leq 1 \\
& x \text{ free}
\end{aligned}
$$

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a) Without constructing the dual program directly, use strong and/or weak duality to draw a conclusion about (the existence of) optimal solutions of the dual program.

b) Construct the dual program, and solve it. To solve it, you may visualize it, use simplex, or use any other valid argument.

exercise	1	2	3	4	5	6
points	8	8	8	8	8	8