Test Linear Optimization

Monday, January 18, 2021, 9.00-12.00

Give motivations for all your answers. You are not allowed to use a calculator.

exercise	1	2	3	4	5	6
points	8	8	8	8	8	8

1. A restaurant owner has D units of dairy, M units of meat, and F units of flour available.
   He can use these ingredients to create k different dishes $\{1,\ldots, k\}$.
   For each dish $i \in \{1, \ldots, k\}$, one unit of dish i requires d_i units of dairy, m_i units of meat, and f_i units of flour, and it sells for p_i.
   Due to demand, the owner can sell at most u_i units of dish i.
   Any unused dairy and meat becomes worthless. But unused flour can be stored and is worth v dollars per unit.

   a) Construct a linear program that maximizes the owner's profit (including the value of unused flour).

   b) Without computing the dual, what can you say about the shadow price of flour (in terms of the parameters)? Intuitively explain why.

2. Consider two **convex** sets $A, B \subseteq \mathbb{R}^n$.
   Also consider sets $P = \{\mathbf{p} \mid \mathbf{p} = \mathbf{a} + \mathbf{b}, \mathbf{a} \in A, \mathbf{b} \in B\}$ and
   $Q = \left\{\mathbf{q} \mid \mathbf{q} = \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix}, \mathbf{a} \in A, \mathbf{b} \in B\right\}$.

   For example, if $\begin{bmatrix} 1 \\ 2 \end{bmatrix} \in A$ and $\begin{bmatrix} 3 \\ 4 \end{bmatrix} \in B$, then $\begin{bmatrix} 4 \\ 6 \end{bmatrix} \in P$ and $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \in Q$.

   For each of the following statements, either give a proof or a counterexample.

   S1) P is convex.

   S2) Q is convex.

3. Consider the following two equivalent descriptions of the same polyhedron:
   $P = \{\mathbf{x} \mid A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ and $Q = \{\mathbf{x} \mid A\mathbf{x} \geq \mathbf{b}, A\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$.
   Prove the following statements, using the definition of basic feasible solution (not the definition of vertex or extreme point):

   a) Any basic feasible solution described by P is a basic feasible solution described by Q.

   b) Any basic feasible solution described by Q is a basic feasible solution described by P.

   c) Except for $\mathbf{x} = \mathbf{0}$, any basic feasible solution described by Q is degenerate.

Turn the page for exercises 4, 5 and 6!

4. Solve the following linear program using the simplex method.

$$\begin{array}{llll} \min & -6x_1 & +4x_2 & \\ \text{s.t.} & -3x_1 & +8x_2 & \leq 3 \\ & 3x_1 & -6x_2 & \leq 1 \\ & 4x_1 & -9x_2 & \leq 2 \\ & \multicolumn{3}{l}{x_1 \geq 0, x_2 \geq 0} \end{array}$$

5. a) Consider the following linear program.

$$\begin{array}{lllll} \max & 3x_1 & +x_2 & & \\ \text{s.t.} & 2x_1 & +x_2 & +x_3 & = 1 \\ & x_1 & -2x_2 & +3x_3 & = -2 \\ & \multicolumn{4}{l}{x_1 \geq 0, x_2 \geq 0, x_3 \geq 0.} \end{array}$$

Construct the corresponding phase 1 linear program for the two-phase simplex method. (You don't have to solve it.)

b) Recall that the lexicographic pivoting rule works as follows: When column j leaves the basis, we divide all the rows with strictly positive entries u_i in column j by u_i, When among these rows, row i is the lexicographically smallest, the i-th basic variable leaves the basis.

Suppose that among these rows, we select the lexicographicaly largest row instead. Which of the following statements about this pivoting rule is true? Motivate your answer.

S1) The rule does not work, since it could happen that the solution after some step is not a vertex.

S2) After each step, the solution is a vertex, but the rule does not work, as the top row does not represent the reduced costs.

S3) After each step, the solution is a vertex, and the top row represents the reduced costs, but this does not prevent cycling, so using this rule we might never attain the optimum.

S4) This rule works perfectly and it prevents cycling.

6. Consider the following linear program with optimal solution $x_1 = 2, x_2 = -1, x_3 = 0$.

$$\begin{array}{lllll} \max & 3x_1 & -x_2 & +x_3 & \\ \text{s.t.} & x_1 & +x_2 & +3x_3 & \leq 1 \\ & x_1 & +2x_2 & -x_3 & \leq 2 \\ & x_1 & -x_2 & +x_3 & \leq 3 \\ & \multicolumn{4}{l}{x_1 \geq 0, \underline{x_2 \leq 0}, x_3 \geq 0} \end{array}$$

a) Construct the dual program.

b) Use complementary slackness to compute an optimal solution of the dual program.

c) Use a corollary of weak duality to verify your answer to part b).