Test Linear Optimization
Monday, January 24, 2022, 8.45-11.45

Give motivations for all your answers. You are not allowed to use a calculator.

1. You have L m^2 of land available.
   You have a budget of B dollars to buy trees to plant on your land.
   There are n different types of trees available.
   A tree of type i costs d_i dollars, it requires l_i m^2 of land, and it produces o_i liters of oxygen per year.

   a) Construct a linear program that maximises the yearly oxygen production of your land.

   b) Describe in words the interpretation of the shadow price (the value of the variable in an optimal dual solution) corresponding to your budget-constraint.

2. **Prove** that the set $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ is convex. You may use that $(a^2 + b^2 \leq 1$ and $c^2 + d^2 \leq 1)$ implies that $ac + bd \leq 1$ without proving it.

3. Let $P \subseteq \mathbb{R}^2$ be a polyhedron. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear operator, and let $G : \mathbb{R}^2 \to \mathbb{R}^2$ be an invertible linear operator.
   Recall that $T(P)$ denotes the set of all $\mathbf{v} \in \mathbb{R}^2$ for which $\mathbf{v} = T(\mathbf{w})$ for some $\mathbf{w} \in P$.
   For each of the following statements, either give a proof or a counterexample.

   S1) For any vertex $\mathbf{v}$ of P, $F(\mathbf{v})$ is a vertex of $F(P)$.

   S2) For any vertex $\mathbf{v}$ of P, $G(\mathbf{v})$ is a vertex of $G(P)$.

4. Solve the following linear program using the simplex method.

$$
\begin{array}{llllll}
\min & -3x_1 + 2x_2 + x_3 & & & & \\
\text{s.t.} & x_1 & +2x_2 & & \leq 4 \\
 & 3x_1 & -3x_2 & +x_3 & \leq 3 \\
 & & 4x_2 & +2x_3 & \leq 2 \\
 & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0 & & & &
\end{array}
$$

Turn the page for exercises 5 and 6.

5. Consider a standard form LP

$$\min \mathbf{c}'\mathbf{x}$$
$$\text{s.t.} \quad A\mathbf{x} = \mathbf{b}$$
$$\mathbf{x} \geq \mathbf{0}$$

with 2 identical columns: $A_i = A_j$ for some i, j. Let $c_j < c_i$. For each of the following statements, either give a proof or a counterexample.

S1) For any basic feasible solution $\mathbf{x}$ for which x_i is a basic variable, the reduced cost $\bar{c}_j$ equals $c_j - c_i$.

S2) In any step of the simplex method, at a basic feasible solution $\mathbf{x}$ where x_j is a basic variable, x_i does not become basic in the next step.

6. a) Determine the dual of the linear program

$$\max x - y$$
$$\begin{array}{lrrrl} \text{s.t.} & x & & -z & = 2 \\ & & 2y & +z & \leq 1 \end{array}$$
$$x \geq 0, y \leq 0, z \text{ free}$$

b) Consider the following LPs:

LP1	LP2
$\max \mathbf{c}'\mathbf{x}$	$\max \mathbf{c}'\mathbf{x}$
s.t. $\mathbf{A}\mathbf{x} = \mathbf{b}$	s.t. $\mathbf{A}\mathbf{x} = \mathbf{b}$
	$\mathbf{A}'\mathbf{p} = \mathbf{c}$
	$\mathbf{c}'\mathbf{x} = \mathbf{b}'\mathbf{p}$

Prove that if LP1 has an optimal solution, then LP2 has an optimal solution with the same objective value.

exercise	1	2	3	4	5	6
points	8	8	8	8	8	8