Test Linear Optimization

Wednesday, January 31, 2024, 8.45-11.45

Give motivations for all your answers. You are not allowed to use a calculator.

1. We have a set of foods F and a set of vitamins V.
   Any unit of food $i \in F$ contains a_{ij} units of vitamin $j \in V$ and c_i calories, and costs d_i dollars.
   To stay healthy, for each vitamin j, we need at least u_j units per day.
   We have a budget of b dollars per day.

   a) Construct a linear program that finds the diet with the smallest amount of calories among all healthy diets that fit the budget.

   b) Suppose that, instead of finding the healthy diet with the smallest amount of calories, we are interested in a healthy diet that contains an amount of calories c as close to some target c^* of calories per day as possible. That is, we want to minimize $|c - c^*|$, where c denotes the amount of calories in our diet. Construct a linear program that finds such a diet.

2. Consider a polyhedron $P \subseteq \mathbb{R}^n$, vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, and a set $S = \{\mathbf{x} \mid \mathbf{x} = \mathbf{a} + \lambda \mathbf{b}, \lambda \in \mathbb{R}\}$. Prove that S contains at most two vertices of P.

3. Consider the following linear program:

$$\begin{array}{lrrl} \max & x & & \\ \text{s.t.} & x & +y & \geq 4 \\ & x & -2y & \geq -6 \\ & x & -y & \leq 6 \end{array}$$

   a) Use Fourier-Motzkin to determine all objective values that correspond to feasible solutions.

   b) Use Fourier-Motzkin to determine an optimal solution.

   You do **not** get points for using any method other than Fourier Motzkin.

4. Solve the following linear program using the simplex method.

$$\begin{array}{lrrl} \min & -3x_1 & +2x_2 & \\ \text{s.t.} & 3x_1 & -6x_2 & \leq 1 \\ & 4x_1 & -9x_2 & \leq 2 \\ & -3x_1 & +8x_2 & \leq 3 \\ & \multicolumn{3}{l}{x_1 \geq 0, x_2 \geq 0} \end{array}$$

Turn the page for exercises 5, 6, and 7!

5. Consider the following tableau.

-3	1	a	0	0
b	1	c	1	0
2	d	4	0	1

a) Find all values for a, b, c, d for which the simplex method takes at least one more step, but in the next step, the objective value does not change.

b) Find all values for a, b, c, d for which this can be the initial tableau in phase 1 of the 2-phase simplex method.

6. Consider the following linear program.

$$\begin{array}{lrrl} \max & x_1 & +x_2 & \\ \text{s.t.} & 2x_1 & +x_2 & \leq 1 \\ & 5x_1 & -x_2 & \leq 0 \end{array}$$
$x_1 \geq 0, x_2$ free

a) Construct its dual program, using the definition of dual program.

b) Determine a linear program that is equivalent to the primal program, and for which all variables have a non-negativity constraint.

c) Construct the dual program of the LP found in b), using the definition of dual program.

d) Give a short reason that the duals found in parts a) and c) are equivalent (without referring to the result that equivalent primals always yield equivalent duals).

7. Let $A, B \in \mathbb{R}^{m \times n}$. Let $\mathbf{b}, \mathbf{d} \in \mathbb{R}^n$. Consider the following two statements.

S1) There exists some vector $\mathbf{x}$ for which $A\mathbf{x} = \mathbf{b}$ and $B\mathbf{x} \leq \mathbf{d}$.

S2) There exist some vectors $\mathbf{u}, \mathbf{v}$ for which:
$\mathbf{v} \geq \mathbf{0}, \mathbf{u}'A + \mathbf{v}'B = \mathbf{0}'$, and $\mathbf{u}'\mathbf{b} + \mathbf{v}'\mathbf{d} < \mathbf{0}$.

a) Prove that at most one of these statements is true.

b) Prove that at least of of these statements is true by constructing the dual of the following LP:

$$\begin{array}{ll} \max & 0 \\ \text{s.t.} & A\mathbf{x} = \mathbf{b} \\ & B\mathbf{x} \leq \mathbf{d} \end{array}$$

exercise	1	2	3	4	5	6	7
points	6	6	6	6	6	6	6