Faculty of Electrical Engineering, Mathematics and Computer Science
Applied Finite Elements, Mastermath
EXAM APRIL 2016

1 Given Holand & Bell's Theorem for a line segment and for a triangle:

Theorem 1: Let be be a line segment in $\mathbb{R}^2$ with vertices (x_1,y_1) and (x_2,y_2), and let $\lambda_1(x,y)$ and $\lambda_2(x,y)$ be linear on be, for which

$$\lambda_i(x_j,y_j) = \delta_{ij}, \qquad \text{where } \delta_{ij} \text{ represents the Kronecker Delta,}$$

and let $m_1, m_2 \in \mathbb{N} = \{0,1,2,\ldots\}$, then

$$\int_{be} \lambda_1^{m_1}\lambda_2^{m_2}d\Gamma = \frac{|be|m_1!m_2!}{(1+m_1+m_2)!}, \qquad \text{where } |be| \text{ denotes the length of line segment } be. \tag{1}$$

Theorem 2: Let e be a triangle in $\mathbb{R}^2$ with vertices (x_1,y_1), (x_2,y_2) and (x_3,y_3), and let $\lambda_1(x,y)$, $\lambda_2(x,y)$ and $\lambda_3(x,y)$ be linear on e, for which

$$\lambda_i(x_j,y_j) = \delta_{ij}, \qquad \text{where } \delta_{ij} \text{ represents the Kronecker Delta,}$$

and let $m_1, m_2, m_3 \in \mathbb{N} = \{0,1,2,\ldots\}$, then

$$\int_{e} \lambda_1^{m_1}\lambda_2^{m_2}\lambda_3^{m_3}d\Omega = \frac{|\Delta_e|m_1!m_2!m_3!}{(2+m_1+m_2+m_3)!}, \qquad \text{where } \frac{|\Delta_e|}{2} \text{ represents the area of triangle } e. \tag{2}$$

In this assignment, the λ-functions are always linear and always satisfy $\lambda_i(x_j,y_j) = \delta_{ij}$.

a Show that the Newton-Cotes numerical integration rule using linear functions over a line-segment be with vertices (x_1,y_1) and (x_2,y_2) is given by

$$\int_{be} g(x,y)d\Gamma \approx \frac{|be|}{2}(g(x_1,y_1)+g(x_2,y_2)). \tag{3}$$

(1 pt)

b Show that the Newton-Cotes numerical integration rule using linear functions over triangle e with vertices (x_1,y_1), (x_2,y_2) and (x_3,y_3) is given by

$$\int_{e} g(x,y)d\Gamma \approx \frac{|\Delta_e|}{6}\sum_{p=1}^{3} g(x_p,y_p). \tag{4}$$

(1 pt)

c Next, we consider quadratic basisfunctions over triangle e with vertices (x_1,y_1), (x_2,y_3) and (x_3,y_3), and midpoints (x_4,y_4), (x_5,y_5) and (x_6,y_6) on the faces of e. For the quadratic functions, we use the following basisfunctions

$$\phi_i(x,y) = \lambda_i(x,y)(2\lambda_i(x,y)-1), \text{ for } i \in \{1,2,3\},$$

and

$$\phi_4(x,y) = 4\lambda_1(x,y)\lambda_2(x,y),\ \phi_5(x,y) = 4\lambda_2(x,y)\lambda_3(x,y),\ \phi_6(x,y) = 4\lambda_3(x,y)\lambda_1(x,y).$$

i Show that $\phi_i(x_j,x_j) = \delta_{ij}$ for $i,j \in \{1,\ldots,6\}$. (1 pt)

ii Show that the Newton-Cotes numerical integration using quadratic basis functions over triangle e is given by

$$\int_{e} g(x,y)d\Gamma \approx \frac{|\Delta_e|}{6}\sum_{p=4}^{6} g(x_p,y_p). \tag{5}$$

(2 pt)

2 Given the following functional, where $u(x,y)$ is subject to an essential boundary condition

$$J[u] = \int_{\Omega} \sqrt{1+|\nabla u|^2}\, d\Omega,$$

$$u(x,y) = u_0(x,y), \qquad \text{on } \partial\Omega,$$

where Ω is a bounded domain in $\mathbb{R}^2$ with boundary $\partial\Omega$. We are interested in the minimiser for the above functional:

Find u, subject to $u = u_0(x,y)$ on $\partial\Omega$ such that $F(u) \le F(v)$ for all v subject to $v = u_0(x,y)$ on $\partial\Omega$.

a Derive the Euler-Lagrange equation (PDE) for $u(x,y)$. (2 pt)

b Derive the Ritz equations. (1 pt)

c We approximate the solution to the minimisation problem by Ritz' Method.

i Describe how you would use Picard's method to approximate the solution to the above problem. (2 pt)

ii Give the element matrix based on linear triangular elements. You may use $|\Delta|$ and $\phi_i = \alpha_i + \beta_i x + \gamma_i y$ for the basis functions. (2 pt)

3 We consider the following boundary value problem for $u = u(t,(x,y))$ to be determined in $\Omega = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ (bounded by $\partial\Omega$) contained in the unit circle:

$$\begin{cases} \nabla \cdot [\mathbf{v}u - D\nabla u] = f(x,y), & \text{in } \Omega, \\ \mathbf{v}(x,y)\cdot \mathbf{n}u - D\dfrac{\partial u}{\partial n} = g(x,y), & \text{on } \partial\Omega, \end{cases} \tag{6}$$

Here $\mathbf{v}(x,y)$, $f(x,y)$, $g(x,y)$ are given functions and $D > 0$ is a constant.

a Derive the compatibility condition for f and g. (1 pt)

b Derive the weak formulation in which the order of spatial derivatives is minimized. *Hint:* apply partial integration on both terms and keep the terms between the brackets as one expression. (2 pt)

c Derive the Galerkin Equations to the weak form in part b. (1 pt)

d We use linear triangular elements to solve the problem. All answers may be expressed in terms of the coefficients in the equations and in the coefficients in $\phi_i = \alpha_i + \beta_i x + \gamma_i y$.

i Compute the element matrix and element vector for an internal triangle. Hint: Use Newton–Cotes integration. (2 pt)

ii Compute the element matrix and element vector for a boundary element. Hint: Use Newton–Cotes integration. (2 pt)

$$\text{Exam Grade} = \frac{\text{Sum over all credits}}{2}.$$