Answers Exam AFE May 22, 2017

① a Let $\varphi_i(\underline{x}_j) = \delta_{ij}$, and let $\varphi_i(\underline{x}) = a_i + b_i x + c_i y + d_i xy$,

with $\underline{x}_1 = (1,0)$, $\underline{x}_2 = (0,1)$, $\underline{x}_3 = (-1,0)$, $\underline{x}_4 = (0,-1)$.

Then

$$(i)\quad \left.\begin{aligned} \varphi_1(\underline{x}_1) = 1 &\Rightarrow a_1 + b_1 = 1 \\ \varphi_1(\underline{x}_2) = 0 &\Rightarrow a_1 + c_1 = 0 \\ \varphi_1(\underline{x}_3) = 0 &\Rightarrow a_1 - b_1 = 0 \\ \varphi_1(\underline{x}_4) = 0 &\Rightarrow a_1 - c_1 = 0 \end{aligned}\right\}$$

From these equations, we see that d_i is (ii) not determined. Moreover it can be seen immediately that this system is inconsistent.

Hence the form $\varphi_i(x,y) = a_i + b_i x + c_i y + d_i xy$ cannot be used.

b Let $\tilde{e} = (0,1) \times (0,1)$.

(i) From interpolation we have

$$g(s,t) \approx \sum_{j=1}^{4} g(s_j,t_j)\,\phi_j(s,t),$$

where
$$\phi_1(s,t) = (1-s)(1-t)$$
$$\phi_2(s,t) = s(1-t)$$
$$\phi_3(s,t) = \cancel{(1-s)t}\ st$$
$$\phi_4(s,t) = (1-s)t.$$

Newton-Cotes resides on the principle to use

$$\int_{\tilde{e}} g(s,t)\,d\Omega_{st} \simeq \sum_{j=1}^{4} g(s_j,t_j) \int_{\tilde{e}} \phi_j(s,t)\,d\Omega_{st}.$$

Since $\tilde{e}$ is a square, we can write

$$\int_{\tilde{e}} (1-s)(1-t)\,ds\,dt = \int_0^1\!\!\int_0^1 (1-s)(1-t)\,ds\,dt = \int_0^1 (1-s)\,ds \int_0^1 (1-t)\,dt$$

$$= \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}.$$ Note that $\int_0^1 (1-t)\,dt = -\int_1^0 u\,du = \int_0^1 u\,du = \int_0^1 t\,dt.$

Hence

$$\int_{\tilde{e}} \phi_i(s,t)\, ds\, dt = \frac{1}{4}, \quad i = 1,2,3,4.$$

Herewith we get

$$\int_{\tilde{e}} g(s,t)\, d\Omega_{st} \simeq \frac{1}{4}\left(g(0,0) + g(1,0) + g(1,1) + g(0,1) \right).$$

(ii) We have $\underline{x}(s,t) = \underline{x}_1 + (\underline{x}_2 - \underline{x}_1)s + (\underline{x}_4 - \underline{x}_1)t + (\underline{x}_1 - \underline{x}_2 + \underline{x}_3 - \underline{x}_4)st$

$$\Rightarrow \begin{cases} \dfrac{\partial \underline{x}}{\partial s}(s,t) = \underline{x}_2 - \underline{x}_1 + (\underline{x}_1 - \underline{x}_2 + \underline{x}_3 - \underline{x}_4)t \\[2ex] \dfrac{\partial \underline{x}}{\partial t}(s,t) = \underline{x}_4 - \underline{x}_1 + (\underline{x}_1 - \underline{x}_2 + \underline{x}_3 - \underline{x}_4)s \end{cases}$$

The Jacobian Matrix becomes

$$\mathcal{J} = \begin{bmatrix} \dfrac{\partial x}{\partial s} & \dfrac{\partial x}{\partial t} \\[2ex] \dfrac{\partial y}{\partial s} & \dfrac{\partial y}{\partial t} \end{bmatrix} = \begin{bmatrix} x_2 - x_1 + A_x t & x_4 - x_1 + A_x s \\ y_2 - y_1 + A_y t & y_4 - y_1 + A_y s \end{bmatrix}.$$

The Determinant follows from

$$\frac{\partial(x,y)}{\partial(s,t)} = (x_2 - x_1 + A_x t)(y_4 - y_1 + A_y s) - (y_2 - y_1 + A_y t)(x_4 - x_1 + A_x s).$$

The Jacobian is computed at the four vertices:

$$(s,t) = (0,0) \Rightarrow \frac{\partial(x,y)}{\partial(s,t)} = (x_2 - x_1)(y_4 - y_1) - (y_2 - y_1)(x_4 - x_1).$$

$$(s,t) = (1,0) \Rightarrow \frac{\partial(x,y)}{\partial(s,t)} = (x_2 - x_1)(y_4 - y_1 + A_y) - (y_2 - y_1)(x_4 - x_1 + A_x)$$

$$= (x_2 - x_1)(y_3 - y_2) - (y_2 - y_1)(x_3 - x_2).$$

$$(s,t) = (1,1) \Rightarrow \frac{\partial(x,y)}{\partial(s,t)} = (x_2 - x_1 + A_x)(y_4 - y_1 + A_y) - (y_2 - y_1 + A_y)(x_4 - x_1 + A_x)$$

$$= (x_3 - x_4)(y_3 - y_2) - (y_3 - y_4)(x_3 - x_2).$$

$$(s,t) = (0,1) \Rightarrow \frac{\partial(x,y)}{\partial(s,t)} = (x_2 - x_1)(y_3 - y_2) - (y_2 - y_1)(x_3 - x_2).$$

(iii) $$\mathcal{J}^{-1} = \frac{1}{\det(\mathcal{J})} \cdot \begin{bmatrix} \frac{\partial y}{\partial t} & -\frac{\partial x}{\partial t} \\ -\frac{\partial y}{\partial s} & \frac{\partial x}{\partial s} \end{bmatrix} = \frac{1}{\det(\mathcal{J})} \cdot \begin{bmatrix} y_4 - y_1 + A_y s & x_1 - x_4 - A_x s \\ y_1 - y_2 - A_y t & x_2 - x_1 + A_x t \end{bmatrix}.$$ III

$\underline{x}_1 \overset{\mathcal{J}}{\longleftrightarrow} (s,t) = (0,0)$.

Hence:

$$\mathcal{J}^{-1}(s=0,t=0) = \frac{1}{\det(\mathcal{J})(0,0)} \cdot \begin{bmatrix} y_4 - y_1 & x_1 - x_4 \\ y_1 - y_2 & x_2 - x_1 \end{bmatrix} = \frac{1}{\left(\frac{\partial(x,y)}{\partial(s,t)}(0,0)\right)} \cdot \begin{bmatrix} y_4 - y_1 & x_1 - x_4 \\ y_1 - y_2 & x_2 - x_1 \end{bmatrix}$$

$$= \frac{1}{(x_2-x_1)(y_4-y_1) - (y_2-y_1)(x_4-x_1)} \cdot \begin{bmatrix} y_4 - y_1 & x_1 - x_4 \\ y_1 - y_2 & x_2 - x_1 \end{bmatrix}.$$

2 $\mathcal{J}(u) = \int_\Omega \sqrt{1 + \|\nabla u\|^2} - uf \, d\Omega$ has been given.

$u = ~~g~~ \, u_0(x,y)$ on $\partial\Omega_1$.

a Euler-Lagrange equations are derived from

$$\frac{d}{dt} \mathcal{J}(u + tv)\Big|_{t=0} = 0, \quad \forall v.$$

Since $u = u_0(x,y)$ on $\partial\Omega_1$, we should have

$u + tv = u_0$ on $\partial\Omega_1$ as well, $\forall t \in \mathbb{R}$

Hence $\boxed{v = 0 \text{ on } \partial\Omega_1.}$

$$\mathcal{J}(u+tv) = \int_\Omega \sqrt{1 + \|\nabla u + t \nabla v\|^2} - (u+tv) f \, d\Omega =$$

$$= \int_\Omega \sqrt{1 + \|\nabla u\|^2 + 2t \nabla u \cdot \nabla v + t^2 \|\nabla v\|^2} - (u+tv) f \, d\Omega.$$

$$\Rightarrow \frac{d}{dt} \mathcal{J}(u+tv)\Big|_{t=0} = \int_\Omega \frac{1}{2\sqrt{1 + \|\nabla u\|^2}} \cdot 2 \cdot \nabla u \cdot \nabla v \, d\Omega - \int_\Omega v f \, d\Omega = 0, \quad \forall v.$$

Hence $$\int_\Omega \frac{\nabla u \cdot \nabla v}{\sqrt{1 + \|\nabla u\|^2}} \, d\Omega = \int_\Omega v f \, d\Omega, \quad \forall v, \; v|_{\partial\Omega_1} = 0.$$

Next, we use integration by parts:

$$\int_\Omega \frac{\nabla u \cdot \nabla v}{\sqrt{1+\|\nabla u\|^2}}\, d\Omega = \int_\Omega \nabla \cdot \left[\frac{\nabla u}{\sqrt{1+\|\nabla u\|^2}} \cdot v\right] - \nabla \cdot \left[\frac{\nabla u}{\sqrt{1+\|\nabla u\|^2}}\right] v \, d\Omega = \int_\Omega v f \, d\Omega$$

Use Gauß for the divergence form in the first term to get

$$\int_{\partial\Omega} \frac{1}{\sqrt{1+\|\nabla u\|^2}} \cdot \frac{\partial u}{\partial n} v \, dS - \int_\Omega v \nabla \cdot \left[\frac{\nabla u}{\sqrt{1+\|\nabla u\|^2}}\right] d\Omega = \int_\Omega v f \, d\Omega .$$

Since $v = 0$ on $\partial\Omega_1$, we get

$$\int_{\partial\Omega_2} \frac{v}{\sqrt{1+\|\nabla u\|^2}} \frac{\partial u}{\partial n} \, dS = \int_\Omega v \left(f + \nabla \cdot \left[\frac{\nabla u}{\sqrt{1+\|\nabla u\|^2}}\right]\right) d\Omega , \ \forall v, \ v|_{\partial\Omega_1} = 0 .$$

Restrict v to $v|_{\partial\Omega_2} = 0$, then from application of DuBois-Reymond's Lemma, we get

$$-\nabla \cdot \left[\frac{\nabla u}{\sqrt{1+\|\nabla u\|^2}}\right] = f(x,y) \quad \text{in } \Omega .$$

This implies that we get

$$\frac{1}{\sqrt{1+\|\nabla u\|^2}} \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega_2 \iff \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega_2 .$$

Hence we get the following BVP:

$$\begin{cases} -\nabla \cdot \left[\dfrac{\nabla u}{\sqrt{1+\|\nabla u\|^2}}\right] = f(x,y) & \text{in } \Omega \\ u = u_0(x,y) & \text{on } \partial\Omega_1 \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega_2 \end{cases}$$

b We have $J(u) = \int_\Omega \sqrt{1+\|\nabla u\|^2} - uf \, d\Omega$.

Ritz' Method is based on approximating u by a linear combination of chosen basisfunctions. Note that we have a nonhomogeneous Dirichlet BC, hence we write

$$u(x,y) \simeq u^n(x,y) = \sum_{j=1}^{n} c_j \varphi_j(x,y) + u_B(x,y),$$

where $\varphi_j|_{\partial\Omega_1} = 0, \ \forall j \in \{1,..,n\}$.

Subsequently we set $J(u) \simeq J(u^n)$, and determine the minimizer over the subspace $\mathrm{Span}\{\varphi_1,..,\varphi_n\} \oplus u_B$ by

$$\frac{\partial J(u^n)}{\partial c_i} = 0, \quad \forall i = 1,\dots,n.$$

$$J(u^n) = \int_\Omega \sqrt{1+\|\nabla u^n\|^2} - u^n f \, d\Omega$$

$$\Rightarrow \frac{\partial J(u^n)}{\partial c_i} = \int_\Omega \frac{1}{\sqrt{1+\|\nabla u^n\|^2}} \nabla u^n \cdot \nabla \varphi_i - \varphi_i f \, d\Omega = 0, \quad i \in \{1,\dots,n\}.$$

Hence
$$\sum_{j=1}^{n} c_j \int_\Omega \frac{\nabla\varphi_i \cdot \nabla\varphi_j}{\left(1+\|\sum_{\ell=1}^{n} c_\ell \nabla\varphi_\ell + \nabla u_B\|^2\right)^{1/2}} d\Omega = \int_\Omega \varphi_i f \, d\Omega + $$

$$- \int_\Omega \frac{\nabla u_B \cdot \nabla\varphi_i}{\left(1+\|\sum_{\ell=1}^{n} c_\ell \nabla\varphi_\ell + \nabla u_B\|^2\right)^{1/2}} d\Omega,$$

$i = 1,2,\dots,n$

These are the Ritz equations.

c We use Picard's Fixed Point Method to iteratively approximate the solution to the nonlinear problem.
Let K be the iteration number. Then we determine $u_{K+1}(x,y)$ from $u_K(x,y)$.

From the previous assignment, we have

$$\int_\Omega \frac{\nabla u^n \cdot \nabla \varphi_i}{\sqrt{1+\|\nabla u^n\|^2}}\, d\Omega = \int_\Omega \varphi_i f \, d\Omega,$$

which is nonlinear in u^n. For convenience we drop the superscript n in the above expression. It is advantageous to solve a linear problem at each Picard iteration. Therefore we set

Solve u_{k+1} from $\displaystyle\int_\Omega \frac{\nabla u_{k+1} \cdot \nabla \varphi_i}{\sqrt{1+\|\nabla u_k\|^2}}\, d\Omega = \int_\Omega \varphi_i f \, d\Omega.$

$$u_{k+1}(x,y) = \sum_{j=1}^{n} c_j^{k+1} \varphi_j(x,y) + u_B(x,y). \qquad \left(u_B(x,y) = \sum_{j=n+1}^{n+n_E} u_0(\underline{x}_j)\varphi_j(x,y) \right).$$

Then the coefficients of the right-hand side and of the discretization matrix read as

$$S_{ij}(u_k) = \int_\Omega \frac{\nabla \varphi_i \cdot \nabla \varphi_j}{\sqrt{1+\|\nabla u_k\|^2}}\, d\Omega.$$

$$f_i(u_k) = \int_\Omega \varphi_i f \, d\Omega$$ (not including the boundary part from $\partial\Omega_1$).

Over each element e_p with vertices $(\underline{x}_{p1}, \underline{x}_{p2}, \underline{x}_{p3})$ and basisfunctions $(\varphi_{p1}, \varphi_{p2}, \varphi_{p3})$, we have

$$S_{ij}^{e_p}(u_k) = \int_{e_p} \frac{\nabla \varphi_i \cdot \nabla \varphi_j}{\sqrt{1+\|\nabla u_k\|^2}}\, d\Omega = (\beta_i \beta_j + \gamma_i \gamma_j) \cdot \int_{e_p} \frac{d\Omega}{\sqrt{1+\|\nabla u_k\|^2}}, \quad (*)$$

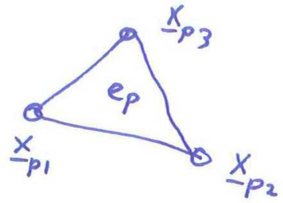
where we used $\varphi_i(x,y) = \alpha_i + \beta_i x + \gamma_i y$.

Further due to the interpolation functions that we use, we have

$$u_k(x,y) \approx \sum u_k(\underline{x}_{p1})\varphi_{p1}(\underline{x}) + u_k(\underline{x}_{p2})\varphi_{p2}(\underline{x}) + u_k(\underline{x}_{p3})\varphi_{p3}(\underline{x}) \text{ in } e_p.$$

$$\Rightarrow \quad \nabla u_h(x,y) = u_h(\underline{x}_{p1}) \begin{bmatrix} \beta_{p1} \\ \gamma_{p1} \end{bmatrix} + u_h(\underline{x}_{p2}) \begin{bmatrix} \beta_{p2} \\ \gamma_{p2} \end{bmatrix} + u_h(\underline{x}_{p3}) \begin{bmatrix} \beta_{p3} \\ \gamma_{p3} \end{bmatrix}.$$

$$\Rightarrow \|\nabla u_h\|^2 = \Big(\sum_{l=1}^{3} u_h(\underline{x}_{pl})\, \beta_{pl}\Big)^2 + \Big(\sum_{l=1}^{3} u_h(\underline{x}_{pl})\, \gamma_{pl}\Big)^2.$$



Note that this is a constant because of using linear elements.

This also implies that the integrand in (*) is constant over e_p.

Hence

$$S_{ij}^{e_p}(u_h) = \frac{(\beta_i \beta_j + \gamma_i \gamma_j) \cdot |\Delta_{e_p}|}{2\left(1 + \Big(\sum_{l=1}^{3} u_h(\underline{x}_{pl})\beta_{pl}\Big)^2 + \Big(\sum_{l=1}^{3} u_h(\underline{x}_{pl})\gamma_{pl}\Big)^2\right)^{1/2}}.$$

Further

$$f_i^{e_p} = \int_{e_p} \varphi_i f \, d\Omega \overset{NC}{\simeq} \frac{|\Delta_{e_p}|}{6} \sum_{l=1}^{3} \underbrace{\varphi_i(\underline{x}_{pl})}_{\delta_{il}} f(\underline{x}_{pl}) =$$

$$= \frac{|\Delta_{e_p}|}{6} f(\underline{x}_{pi}). \qquad i,j \in \{p_1, p_2, p_3\}.$$

Note that there is no contribution from the boundary to the discretisation matrix and right-hand side vector. This follows from the fact that the weak form ~~does not~~ or functional $J(u)$ does not contain any integration over the boundary.

③ a Given

$$\left.\begin{aligned} -\nabla \cdot (D(x,y)\, \nabla u) &= f \quad \text{in } \Omega \\ u &= g \quad \text{on } \partial\Omega \end{aligned}\right\}$$

We derive the weak form:

$$-\int_{\Omega} \varphi\, \nabla \cdot (D(x,y)\nabla u)\, d\Omega = \int_{\Omega} \varphi f \, d\Omega .$$

Integrate by Parts:

$$-\int_{\partial\Omega} \varphi\, D(x,y) \frac{\partial u}{\partial n}\, d\Gamma + \int_{\Omega} D(x,y) \nabla u \cdot \nabla\varphi \, d\Omega = \int_{\Omega} \varphi f \, d\Omega$$

Since $u = g$ on $\partial\Omega \Rightarrow \varphi = 0$ on $\partial\Omega$, we get the following weak form:

$$(W): \begin{cases} \text{Find } u \in H^1(\Omega) \text{ subject to } u|_{\partial\Omega} = g \text{ such that} \\ \int_{\Omega} D(x,y) \nabla u \cdot \nabla \varphi \, d\Omega = \int_{\Omega} \varphi f \, d\Omega, \ \forall \varphi \in H_0^1(\Omega). \end{cases}$$

$(H_0^1(\Omega) := \{\varphi \in H^1(\Omega) \,/\, \varphi|_{\partial\Omega} = 0\}.)$.

b We approximate the solution by a linear combination of a chosen set of basis functions:

$$u(x,y) \simeq u^n(x,y) = \sum_{j=1}^{n} c_j \varphi_j(x,y) + u_B(x,y),$$

$\varphi_j|_{\partial\Omega} = 0$, $j = 1, \dots, n$, $u_B(x,y)$ compensates for the nonhomogeneous Dirichlet BC.

We set $\varphi = \varphi_i$, $i = 1, \dots, n$ to get

$$\sum_{j=1}^{n} c_j \underbrace{\int_{\Omega} D(x,y) \nabla\varphi_i \cdot \nabla\varphi_j \, d\Omega}_{S_{ij}} = \underbrace{\int_{\Omega} \varphi_i f \, d\Omega - \int_{\Omega} D(x,y) \nabla u_B \cdot \nabla \varphi_i \, d\Omega}_{b_i} .$$

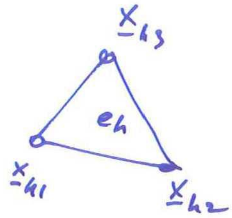
$$\sum_{j=1}^{n} S_{ij} c_j = b_i .$$

c (i) $S_{ij} = \int_\Omega D(x,y)\, \nabla\varphi_i \cdot \nabla\varphi_j \, d\Omega = \sum_{k=1}^{nel} S_{ij}^{e_k}$, where

$$S_{ij}^{e_k} = \int_{e_k} D(x,y)\, \nabla\varphi_i \cdot \nabla\varphi_j \, d\Omega = (\beta_i\beta_j + \gamma_i\gamma_j) \int_{e_k} D(x,y)\, d\Omega,$$

where $\varphi_i = \alpha_i + \beta_i x + \gamma_i y$, $\nabla\varphi_i = \begin{bmatrix}\beta_i \\ \gamma_i\end{bmatrix}$.

We use Newton-Cotes for



$$\int_{e_k} D(x,y)\, d\Omega \simeq \frac{|\Delta_{e_k}|}{6} \sum_{\ell=1}^{3} D(\underline{x}_\ell).$$

Hence $S_{ij}^{e_k} = (\beta_i\beta_j + \gamma_i\gamma_j) \dfrac{|\Delta_{e_k}|}{6} \sum_{\ell=1}^{3} D(\underline{x}_{k\ell})$,

is the element matrix for internal (triangular) elements.

$$f_i^{e_k} = \int_{e_k} \varphi_i f \, d\Omega = \frac{|\Delta_{e_k}|}{6} f(\underline{x}_i), \quad i \in \{k_1, k_2, k_3\}.$$

(ii) There are no boundary elements since the weak form does not contain any integration over boundary segments. Hence there is no boundary element matrix and there is no boundary element vector.

— End of exam AFE May 22, 2017 —