Exam Advanced Linear Programming, May 14, 13.30-16.30

- Switch off your mobile phone, PDA and any other mobile device and put it far away.
- No books or other reading materials are allowed.
- This exam consists of two parts. Part 1 has 6 questions and part 2 has 2 questions. *Write the answers to the different parts on different pieces of exam paper.* Please write down your name on every exam paper that you hand in.
- Answers may be provided in either Dutch or English.
- All your answers should be clearly written down and provide a clear explanation. Unreadable or unclear answers may be judged as false.
- The maximum score per question is given between brackets before the question.

Part 1

(1) *(1 point.)* Formulate Farkas' Lemma.

(**2**) *(1 point.)* Let A be an $m \times n$ matrix, let C be a $k \times n$ matrix and let $b \in \mathbb{R}^m,\ d \in \mathbb{R}^k$. Prove that exactly one of the following holds:
(a) there exists an $x \in \mathbb{R}^n$ such that $Ax \leq b$, $Cx = d$, and $x \geq 0$; and
(b) there exist $y \in \mathbb{R}^m$, $z \in \mathbb{R}^k$ such that $y \geq 0$, $y^T A + z^T C \geq 0$, and $y^T b + z^T d < 0$.

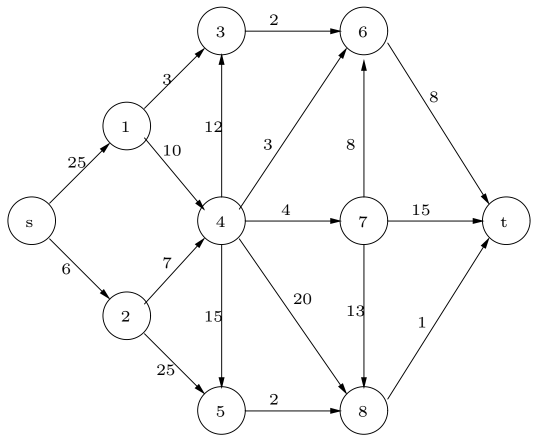
(**3**) *(1 point.)* Give the definition of convex hull. Let $X \subseteq \mathbb{R}^n$ be a finite set and let $y \in \mathbb{R}^n$. Prove that if $y \in \text{conv.hull}(X)$, then there exists a set $X' \subseteq X$ such that $|X'| \leq n+1$ and $y \in \text{conv.hull}(X')$.

(**4**) *(1 point.)* Consider the linear optimization problem

$$\max \left\{ \begin{bmatrix} -1 & 3 & 2 \end{bmatrix} x \;\middle|\; \begin{bmatrix} -1 & 1 & 2 \\ -3 & 2 & 1 \\ 8 & -3 & 2 \end{bmatrix} x \leq \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, x \geq 0 \right\}.$$

(a) Solve this problem using the simplex method.
(b) Determine the dual of this problem.
(c) State the complementary slackness condition for optimality of a primal feasible solution x and a dual feasible solution y.
(d) Find a dual optimal solution.

(**5**) *(1 point.)* Consider the network given in the figure below. The number at an arc represents the capacity of that arc.



Determine an s–t cut of minimum capacity in this network. Give a short argument why your answer is correct.

(**6**) *(1 point.)* (Duality and the max-flow min-cut theorem.) Consider the maximum flow problem, written as the linear program

$$\begin{array}{rl} \max & \sum_{(s,i)\in\mathcal{A}} f_{si}; \\ \text{s.t.} & \sum_{(i,j)\in\mathcal{A}} f_{ij} - \sum_{(j,i)\in\mathcal{A}} f_{ji} = 0, \ \ \forall i \in \mathcal{N} \setminus \{s,t\}; \\ & 0 \le f_{ij} \le u_{ij}, \ \ \forall (i,j) \in \mathcal{A}. \end{array}$$

(a) Let p_i be a price variable associated with the flow conservation constraint at node i. Let q_{ij} be a price variable associated with the capacity constraint at arc (i,j). Write down a minimization problem, with variables p_i and q_{ij}, whose dual is the maximum flow problem.

(b) Show that the optimal value in the minimization problem is equal to the minimum cut capacity, and prove the max-flow min-cut theorem.

Part 2

(1) *(2 points)* We consider the resource constrained shortest path problem. We are given a directed graph (V, A), where V is the set of nodes and A is the set of arcs. Each $(i, j) \in A$ has a cost c_{ij} and a traversal time t_{ij}. Moreover, there is a source node $s \in V$ and a sink $t \in V$. The objective is to find a path from s to t which has minimal cost and is such that the total traversal time does not exceed the maximum T_{max}. This problem can be formulated as integer linear programming problem as follows:

$$
\begin{array}{lcll}
\min & \sum_{(i,j)\in A} c_{ij}x_{ij} & & \\
\text{s.t.} & & & \\
\sum\limits_{j:(s,j)\in A} x_{sj} & = & 1 & \\
\sum\limits_{j:(j,t)\in A} x_{jt} & = & 1 & \\
\sum\limits_{i:(i,j)\in A} x_{ij} & = & \sum\limits_{k:(j,k)\in A} x_{jk} & \forall j \in V \setminus \{s,t\} \\
\sum\limits_{(i,j)\in A} t_{ij}x_{ij} & \leq & T_{\max} & \\
x_{ij} & \in & \{0,1\} & \forall (i,j) \in A
\end{array}
$$

(a) Write the Lagrangean relaxation subproblem that you obtain by dualizing the constraint on the maximal total traversal time.

(b) Let Z_{IP} be the optimal value of the original ILP formulation and $Z(\lambda)$ be the optimal value of the Lagrangean subproblem with multiplier λ. Do we have $Z(\lambda) \leq Z_{IP}$ or $Z(\lambda) \geq Z_{IP}$? Prove your answer.

(c) Show that the constraint matrix of the Langrangean subproblem is Totally Unimodular.

(d) What is the implication of (c) for the optimal value of the Lagrangean Dual?

(2) *(2 points)* A large production company owns m distribution centers from which goods are sent to the customers. For each center i ($i = 1, \ldots, m$) we know the cost k_i of keeping the center open, its capacity M_i, and the transportation cost q_i per unit for transporting goods from the production facility to center i. For each customer j ($j = 1, \ldots, n$) we know his demand v_j and the transportation cost c_{ij} per unit for sending goods from center i to customer j. Each customer has to be served by exactly one distribution center.

(a) The company wants to find the optimal delivery plan for the current situation in which all distribution centers are open. Find an integer linear programming formulation for this problem with a **polynomial** number of variables (do not use the formulation from part (c)).

(b) To achieve a budget reduction, the company considers closing some of the distribution centers. Extend the model of part (a) to an integer linear programming formulation to decide which distributions centers should be kept open if the company wants to minimize the distribution cost.

(c) An alternative way to formulate the problem in (b) is by using **customer groups**. A customer group for center i is a set of customers served by distribution center i such that the capacity M_i of center i is not exceeded. Define S_i as the collection of all feasible customers groups for center i. Give an integer linear programming formulation for the problem of part (b) based on customer groups.

(d) Describe how the LP-relaxation of this formulation can be solved by column generation. Your description should include a formulation of the pricing problem for a given center i. You do not have to describe how to solve the pricing problem (but you are allowed to do so).