# Examination

## Advanced Linear Programming
## Spring Semester 2021

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## *Information*

### General Information

- The examination lasts 180 minutes.
- You are allowed to use any kind of physical or digital material to assist you during the exam. Digital material has to be downloaded/prepared before the examination starts and has to be accessed using the mouse/touch pad of your computer.
- You are not allowed to communicate with other people during the exam (neither physically nor digitally).
- Pocket or online calculators are not allowed.
- Prepare your solutions using pen and paper. No digitally prepared answers are accepted.
- Immediately after completing (and uploading) the exam, the lecturers may contact a random sample of students over video-conferencing, and may ask them a short follow-up question on one of the problems. This is to detect fraud and will not contribute to the final grade.
- After grading, the examiners may ask some students to explain their solution over video-conferencing in order to check the authenticity of their work.

### Statement of Integrity

- You have to write and sign the following statement of integrity.

> This exam will be solely undertaken by myself, without any assistance from others, and without use of sources other than those allowed (all physical written material and digital material that has been downloaded before the exam starts).

### Submission

- Prepare scans of the statement of integrity and your solutions, e.g., using your phone.
- Merge your scans into a single PDF.
- Upload your scans to ELO within 15 minutes after the end of the exam.
- After submitting your solutions, do not eliminate your answers. In case that the quality of your scans is not sufficient, we may ask for alternative scans with a higher resolution. Based on your initial submission, we will compare both versions and check whether they match.

***GOOD LUCK***

## *Part 1*

**Exercise 1** 1 point

Let $P$ be a bounded polyhedron in $\mathbb{R}^n$, let $a$ be a vector in $\mathbb{R}^n$, and let $b$ be some scalar. We define $Q = \{x \in P \mid a'x = b\}$. Show that every extreme point of $Q$ is either an extreme point of $P$ or a convex combination of two adjacent extreme points of $P$.

**Solution:** During class we have proven that an extreme point is the same as a basic feasible solution. Thus if $x^*$ is an extreme point of $Q$ then there must be $n$ linearly independent constraints that are met with equality in $x^*$. One of the equalities is $a'x^* = b$. We distinguish two cases.

In the first case next to that $a'x^* = b$ $n$ linearly independent constraints of $P$ are tight. But then $x^*$ is also an extreme point of $P$.

In the second case next to $a'x^* = b$ $n-1$ constraints of $P$ must be tight that are linearly independent among themselves and also from $a'x = b$. The intersection of the $n-1$ constraints from $P$ define a line through $x^*$, say $x^* + \lambda d$. Since $P$ is bounded, walking away from $x^*$ along this line in one direction we will hit a constraint that was not tight in $x^*$ and that is linearly independent from the other $n-1$ constraint. So together with them define an extreme point, say $y$ of $P$. Similarly, walking away from $x^*$ in the other direction will arrive at an extreme point, say $z$ of $P$. But then $x^*$ is a convex combination of $y$ and $z$.

**Exercise 2** 0.5+0.5 points

1. Let $A$ be an $m \times n$ matrix, let $u \in \mathbb{R}^n$, $u > 0$, and let $b \in \mathbb{R}^m$. Fill in (b) in the following statement: Exactly one of the following holds:
   (a) there exist $x \in \mathbb{R}^n$ such that $Ax = b$ and $0 \leq x \leq u$;
   (b)
2. Prove the statement.

**Solution:** $\exists p \in \mathbb{R}^m, q \in \mathbb{R}^n$: $p^T A + q^T \geq 0$, $q^T \geq 0 \wedge p^T b + q^T u < 0$

$\forall$ $p \in \mathbb{R}^m$ such that $p^T A \geq 0$ we have $p^T b > 0$ there exist $x \in \mathbb{R}^n$ such that $Ax = b$ and $0 \leq x \leq u$;

We can rewrite statement [a] as
[a'] there exist $x, s \in \mathbb{R}^n$ such that $Ax = b$ and $Ix + Is = u$, $0 \leq x$, $s \geq 0$;

Farkas Lemma says that only one of the following is true:

(a) there exist $x \in \mathbb{R}^n$ such that $Ax = b$ and $0 \leq x \leq u$;

(b) $\exists p \in \mathbb{R}^m$: $p^T A \geq 0 \wedge p^T b < 0$

Applying Farkas Lemma on our [a'] gives for the alternative that

$\forall p \in \mathbb{R}^m, q \in \mathbb{R}^n$ such that $p^T A + q^T I \geq 0$ and $q^T I \geq 0$ we have $p^T b + q^T u > 0$,

which translates directly to our statement [b]

**Exercise 3** 0.5+0.5 points

While solving a standard form linear programming problem, we arrive at the following tableau, with $x_3$, $x_4$ and $x_5$ being the basic variables.

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | |
|---|---|---|---|---|---|---|
| $(\bar{c})$ | $\alpha$ | $-2$ | 0 | 0 | 0 | $-10$ |
| $(x_3)$ | $-1$ | $\delta$ | 1 | 0 | 0 | 4 |
| $(x_4)$ | $\beta$ | $-4$ | 0 | 1 | 0 | 1 |
| $(x_5)$ | $\gamma$ | 3 | 0 | 0 | 1 | $\eta$ |

The entries $\alpha$, $\beta$, $\gamma$, $\delta$, $\eta$ are in the tableau are unknown parameters. For each one of the following statements, find some parameter values that will make the statement true.

(a) The current solution is optimal and there are multiple optimal solutions.

(b) The optimal cost is $-\infty$.

**Solution: Answer Part A.** For feasibility $\eta \geq 0$. For optimality $\alpha \leq 0$. For multiple optima $\alpha = 0$. The rest can be chosen. Any values satisfying these restrictions give a correct answer.

**Answer Part B.** Any solution with $\alpha > 0,\ \beta \leq 0\ and\ \gamma \leq 0$ and $\eta \geq 0$ (for feasibility).

**Exercise 4** 1.5+0.5 points

Consider the max-flow problem on the network in Figure 1. The number next to an arc gives the capacity of that arc.

a) Suppose that the first path found by the Ford-Fulkerson algorithm is the path $s, 1, 4, 7, t$, which has bottleneck capacity 4. You have to start with this path and complete the exercise, finding the maximum flow from $s$ to $t$ using Ford-Fulkerson. Show clearly which flow augmenting paths you choose in each iteration.
*NB: Not starting with the given first path will give no points at all!!!*

b) Using the residual network in the optimum, find a minimum $s$-$t$-cut. Indicate clearly how you find the cut.

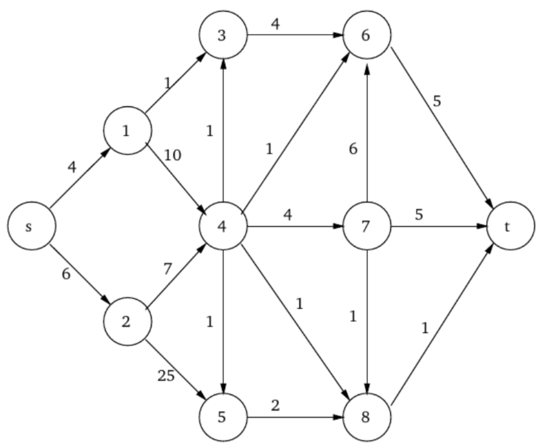



Figure 1: Network

**Solution: Indication of an Answer Part A.** The optimal flow has size 8, obtained by 4 augmenting paths after the given one. For the last augmenting path the residual network must be constructed.

**Answer Part B.** The minimum cut is given by the set of nodes reachable from $s$ in the residual network corresponding to the maximum flow. The cut should be $\{s, 1, 2, 4, 5, 8\}$.

# *Part 2*

**Exercise 5** 0.25+0.25+0.5+0.5 points

Consider the integer program

$$\begin{aligned} \max -x + y& \\ -4x + 2y &\leq 1 \\ -2x + 4y &\leq 11 \\ x + 3y &\leq 17 \\ 2x - y &\leq 6 \\ x, y &\text{ integral.} \end{aligned}$$

(a) Draw the feasible region of the LP relaxation and mark all solutions of the integer program. If the feasible region is unbounded, draw the the part containing all vertices and indicate how the region expands towards infinity. Specify which part of your drawing corresponds to the LP relaxation and which to the integer program.

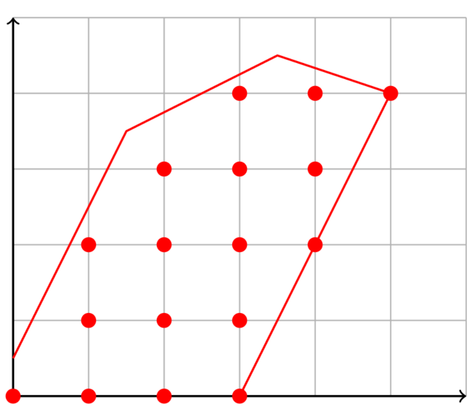
(b) Using your drawing from a, find an optimal solution of the LP relaxation.

(c) Solve the integer program using branch-and-bound.
*Hint:* You are allowed to solve the LP relaxations graphically.

(d) The point $(x^\star, y^\star) = (\frac{7}{2}, \frac{9}{2})$ is an extreme point of the initial LP relaxation of the above integer program. Derive a Gomory cut for $(x^\star, y^\star)$.

**Solution:**

(a) The picture shows the feasible region of the LP (red lines, open to the bottom left) and the IP (red dots continuing to the bottom left). (0.15 + 0.1 points)



(b) An optimal solution of the LP is given by $(\frac{3}{2}, \frac{7}{2})$.

(c) Since the optimal solution of the LP is not integer, we branch on a fractional variable, e.g., $x$. This gives us the two subproblems with additional constraints $x \leq 1$ (P1) or $x \geq 2$ (P2).

First, we solve P2, which has $(2, \frac{15}{4})$ as optimal solution. It is again not integral, so we branch on $y$, which generates the problems with additional constraints $y \leq 3$ (P2a) and $y \geq 4$ (P2b).

The optimal solution of P2a is $(2, 3)$ with objective value 1; P2b has solution (2.5, 4) with value 1.5.

Second, we solve P1, which has $(1, \frac{5}{2})$ as optimal solution. It is again not integral, so we branch on $y$, which generates the subproblems with additional constraint $y \leq 2$ (P1a) or $y \geq 3$ (P1b).

P1b is infeasible and P1a has optimal solution $(\frac{3}{4}, 2)$ with objective value $\frac{5}{4}$. Since the objective function coefficients are all integral, this means that an feasible integer solution at P1a has a value of at most 1. Hence, we can stop branch-and-bound and we return the solution $(2, 3)$ found in subproblem P2a.

(-0.1 points per mistake) (If students use equations instead of inequalities for branching decisions, 0.1. If they correctly classify infeasible solutions, 0.05)

(d) Note that the second and third inequality are satisfied with equality by the given solution. To derive a Gomory cut for a problem in standard form, we consider

$$\begin{aligned} -2x + 4y + s_1 &= 11 \\ x + 3y + s_2 &= 17, \end{aligned}$$

where $s_1$ and $s_2$ are slack variables to turn the initial formulation into standard form. (0.1 points)

Since both inequalities are satisfied with equality, a basis is given by $B = \{1, 2\}$ and the corresponding non-basis is $N = \{3, 4\}$. (0.1 points)

The inverse of the basis matrix is

$$A_B^{-1} = \begin{pmatrix} -2 & 4 \\ 1 & 3 \end{pmatrix} = \frac{1}{10} \begin{pmatrix} -3 & 4 \\ 1 & 2 \end{pmatrix}. \text{ (0.1 points)}$$

Hence,

$$\bar{b} = \frac{1}{10} \begin{pmatrix} -3 & 4 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 11 \\ 17 \end{pmatrix} = \frac{1}{10} \begin{pmatrix} 35 \\ 45 \end{pmatrix},$$

and the two possible Gomory cuts are

$$\begin{pmatrix} x \\ y \end{pmatrix} \leq \begin{pmatrix} \lfloor 3.5 \rfloor \\ \lfloor 4.5 \rfloor \end{pmatrix} - \begin{pmatrix} \lfloor -0.3 \rfloor & \lfloor 0.4 \rfloor \\ \lfloor 0.1 \rfloor & \lfloor 0.2 \rfloor \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} 3 + s_1 \\ 4 \end{pmatrix}. \text{ (0.1 points)}$$

Replacing $s_1$ by $11 + 2x - 4y$ thus leads to the cuts

$$-x + 4y \leq 14 \qquad \text{and} \qquad y \leq 4. \text{ (0.1 points)}$$

**Exercise 6** 0.5+0.75 points

(a) Which of the following three matrices are unimodular and which are totally unimodular? Justify your answers.

$$A = \begin{pmatrix} 1 & -1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

(b) Consider the polyhedron $P \subseteq \mathbb{R}^n \times \mathbb{R}^n$ defined by the inequalities

$$\sum_{j=1}^{i} x_j = y_i, \qquad i \in \{1, \dots, n\}, \tag{1a}$$

$$x_i, y_i \geq 0, \qquad i \in \{1, \dots, n\}. \tag{1b}$$

Prove that all extreme points of $P$ are integral.

**Solution:**

(a) **First matrix:**

$$\det(A) = \det \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = 1 - (1 - 1) = 1.$$

Thus, the matrix is unimodular as it has full row rank. It is not totally unimodular, because the submatrix defined by the first two rows and columns has determinant 2. (0.1 + 0.1)

**Second matrix:** $\det(B) = 2 + 1 - 2 = 1$, i.e., the matrix is unimodular. It is not totally unimodular, because it is not a $\{0, \pm 1\}$-matrix. (0.05 + 0.05)

**Third matrix:** Its determinant is obiously 1. That is, it is unimodular. Also one can readily verify that all submatrices have determinant $\{0, \pm 1\}$. Hence, it is also totally unimodular. (0.1 + 0.1)

(b) The constraint matrix has shape $(A, B)$, where

$$A = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 1 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \dots & 1 & 1 \end{pmatrix}, \qquad B = -E_n.$$

$A$ is totally unimodular since it is an interval matrix (see the exercises). (0.5 points)

Thus, $(A, -I)$ is unimodular by a theorem from the lecture The theorem of Hoffman and Kruskal thus shows that the polyhedron defined by also adding non-negativity constraints is integral. (0.25 points)

**Exercise 7** 2.25 points

In this exercise, we want to solve the linear program

$$\min -2x - 3y \tag{2a}$$
$$x + y + s = 10, \tag{2b}$$
$$x, y, s \geq 0 \tag{2c}$$

using Benders' decomposition. To this end, we keep only the $x$-variable in the master problem and, as discussed in the lecture, we use a new variable $z$ to model the impact of $y$ and $s$ on the objective. Moreover, we assume that we already have some extra knowledge, namely $z \geq -30$. That is, the initial Benders master problem looks like

$$\min -2x + z \tag{3a}$$
$$x \geq 0, \tag{3b}$$
$$z \geq -30. \tag{3c}$$

Find an optimal solution of the linear program (2) using Benders' decomposition with initial master problem (3).

Provide a comprehensible explanation of all your steps, that is, do not just provide formulas, but also provide some text that explains how you read off Benders' cuts from your results or why the method terminates.

**Solution:** To be able to apply Benders' decomposition, we need to find the dual of

$$\min -3y$$
$$y + s = 10 - x$$
$$y, s \geq 0,$$

which is given by

$$\max (10 - x)p$$
$$p \leq -3.$$ (0.25 points)

The auxiliary problem for deriving Benders cuts is thus given by

$$\max (10 - x)p - z\gamma$$
$$p \leq -3\gamma,$$
$$\gamma \geq 0.$$ (0.25 points)

In the first iteration of Benders' decomposition, we derive an optimal solution of the restricted master problem

$$\min -2x + z \tag{4}$$
$$x \geq 0, \tag{5}$$
$$z \geq -30. \tag{6}$$

This problem is unbounded with "solution" $(\infty, 0)$. (0.25 points)

Inserting this solution into the cut generation problem yields

$$\begin{aligned} \max(10-\infty)p& \\ p &\leq -3\gamma, \\ \gamma &\geq 0, \end{aligned}$$

which has $(-1, 0)$ as unbounded ray. Hence, we derive the feasibility cut $(10-x)\cdot(-1) \leq 0$ which is equivalent to $x \leq 10$. (0.25 points)

In the second iteration of Benders' decomposition, we derive an optimal solution of the updated restricted master problem

$$\min -2x + z \tag{7}$$
$$x \leq 10, \tag{8}$$
$$x \geq 0, \tag{9}$$
$$z \geq -30. \tag{10}$$

This problem has optimal solution $(x, z) = (10, -30)$. (0.25 points)

Inserting this solution into the cut generation problem yields

$$\begin{aligned} \max 30\gamma& \\ p &\leq -3\gamma, \\ \gamma &\geq 0, \end{aligned}$$

which has $(-3, 1)$ as unbounded ray. Hence, we derive the optimality cut $(10-x)\cdot(-3) \leq z$ which is equivalent to $3x - 30 \leq z$. (0.25 points)

In the third iteration of Benders' decomposition, we derive an optimal solution of the updated restricted master problem

$$\min -2x + z \tag{11}$$
$$x \leq 10, \tag{12}$$
$$3x - 30 \leq z, \tag{13}$$
$$x \geq 0, \tag{14}$$
$$z \geq -30. \tag{15}$$

This problem has optimal solution $(x, z) = (0, -30)$. (0.25 points)

Inserting this solution into the cut generation problem yields

$$\begin{aligned} \max 10p + 30\gamma& \\ p &\leq -3\gamma, \\ \gamma &\geq 0, \end{aligned}$$

which has $(0, 0)$ as optimal solution. Since the optimal objective value is 0, no violated Benders cut exists. (0.25 points)

The solution $(x, z) = (0, -30)$ is thus optimal. To find a corresponding solution in the original variable space, we solve

$$\begin{aligned} \min -3y& \\ y + s &= 10 \\ y, s &\geq 0, \end{aligned}$$

which yields $y = 10$ and $s = 0$. Thus, $(x, y, s) = (0, 10, 0)$ is an optimal solution of the original problem. (0.25 points)