# Examination

## Advanced Linear Programming
## Spring Semester 2022

**Lecturer:** **Christopher Hojny**
**Leen Stougie**

## *Information*

### General Information

- The examination lasts 180 minutes.
- Switch off your mobile phone, PDA and any other mobile device and put it far away.
- No books or other reading materials are allowed.
- This exam consists of two parts. *Write the answers to the different parts on different pieces of exam paper.* Please write down your name on every exam paper that you hand in.
- Part 1 has 5 questions and part 2 has 3 questions.
- Answers may be provided in either Dutch or English.
- All your answers should be clearly written down and provide a clear explanation. Unreadable or unclear answers may be judged as false.
- The maximum score per question is given between brackets before the question.
- Give your answers to the two parts on separate sheets!!!

*GOOD LUCK*

## *Part 1*

**Exercise 1** 1 point

Formulate Farkas' Lemma.

**Exercise 2** 1 point

Let $A$ be an $m \times n$ matrix, let $C$ be an $m \times k$ matrix and let $b \in \mathbb{R}^m$. Prove that exactly one of the following holds:

(a) there exist $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^k$ such that $Ax + Cu = b$ and $x \geq 0$;

(b) there exist $y \in \mathbb{R}^m$ such that $y^T A \geq 0$, $y^T C = 0$ and $y^T b < 0$.

**Exercise 3** 0.75 points

Consider the simplex method applied to a standard form LP-problem

$$\begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax = b, \\ & x \geq 0, \end{array}$$

with $x \in \mathbb{R}^n$, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ and $A$ a $m \times n$ matrix. Assume that the $m$ rows of the matrix $A$ are linearly independent. For each of the statements that follow, indicate if it is TRUE or FALSE. Argue your answers briefly (you don't need to prove them). A correct answer gives $0.25$ pt., an incorrect answer gives $-0.25$ pt. No answer gives $0$ pt. The total score will always be non-negative.

(i) An iteration of the simplex method may move the feasible solution by a positive distance while leaving the cost unchanged.

(ii) A variable that has just left the basis cannot reenter in the very next iteration.

(iii) A variable that has just entered the basis cannot leave in the very next iteration.

**Exercise 4** 0.75+0.25 points

*Hint: part (b) of this exercise is easier than part (a) and can be made without having part (a) solved correctly.*

Given is the following theorem.

**Theorem 0.1.** *Let $a_1, \ldots, a_m$ be some vectors in $\mathbb{R}^n$, with $m > n + 1$. Suppose that the system of inequalities $a_i^T x \geq b_i$, $i = 1, \ldots, m$, does not have any solutions. Then we can choose $n + 1$ of these inequalities, so that the resulting system of inequalities has no solutions.*

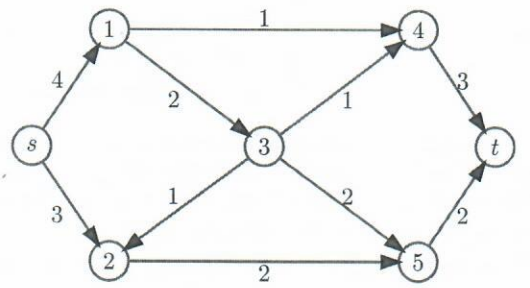
(a) Use this theorem to prove Helly's Theorem.

**Theorem 0.2. Helly's Theorem.** *Let $\mathcal{F}$ be a finite family of polyhedra in $\mathbb{R}^n$ such that every $n + 1$ polyhedra in $\mathcal{F}$ have a point in common. Then all polyhedra in $\mathcal{F}$ have a point in common.*

(b) For $n = 2$, Helly's Theorem asserts that the polyhedra $P_1, P_2, \ldots, P_K$, $(K \geq 3)$ in the plane have a point in common if and only if every three of them have a point in common. Is the result still true with "three" replaced by "two"?

**Exercise 5** 0.5+0.25+0.5 points

(a) Consider the network in the figure. The number next to an arrow gives the capacity of the arrow. Using Ford-Fulkerson we will determine the maximum flow in this network from node $s$ to node $t$.




In the first two iterations of Ford-Fulkerson we have found the following flow augmenting paths:

- $s, 1, 4, t$ with bottleneck capacity 1 because of arrow $(1, 4)$;
- $s, 1, 3, 5, t$ with bottleneck capacity 2 because of arrows $(1, 3), (3, 5)$ and $(5, t)$.

So, we have now a flow of size 3 from $s$ to $t$.

Start from this solution with further iterations of Ford-Fulkerson to find the maximum flow. Also derive the corresponding minimum cut.

(b) (Duality and the max-flow min-cut theorem.) Consider the maximum flow problem, written as the linear program

$$\begin{array}{rl} \max & \sum_{(s,i)\in\mathcal{A}} f_{si}; \\ \text{s.t.} & \sum_{(i,j)\in\mathcal{A}} f_{ij} - \sum_{(j,i)\in\mathcal{A}} f_{ji} = 0,\ \forall i \in \mathcal{N} \setminus \{s, t\}; \\ & 0 \leq f_{ij} \leq u_{ij},\ \forall (i,j) \in \mathcal{A}. \end{array}$$

Let $p_i$ be a price variable associated with the flow conservation constraint at node $i$. Let $q_{ij}$ be a price variable associated with the capacity constraint at arc $(i, j)$. Write down a minimization problem, with variables $p_i$ and $q_{ij}$, whose dual is the maximum flow problem.

(c) Show that the optimal value in the maximization problem is equal to the minimum cut capacity.

## *Part 2*

**Exercise 6** 0.25+0.25+0.6+0.4 points

Consider the integer program

$$\begin{aligned} \max\ & -3x + 2y \\ & -4x + 2y \leq 3 \\ & 3x + 5y \leq 27 \\ & x \leq 4 \\ & x, y \geq 0 \ \text{ and integral.} \end{aligned}$$

(a) Draw the feasible region of the LP relaxation and mark all solutions of the integer program. If the feasible region is unbounded, draw the the part containing all vertices and indicate how the region expands towards infinity. Specify which part of your drawing corresponds to the LP relaxation and which to the integer program.

(b) Using your drawing from a, find an optimal solution of the LP relaxation.

(c) Solve the integer program using branch-and-bound.
*Hint:* You are allowed to solve the LP relaxations graphically.

(d) The point $(x^\star, y^\star) = (0, \frac{3}{2})$ is an extreme point of the initial LP relaxation of the above integer program. Derive a Gomory cut for $(x^\star, y^\star)$.

**Exercise 7** 0.25+0.25+0.7 points

Consider the linear program

$$\begin{aligned} \min\ & -2x_1 - x_2 - 3x_3 \\ & 2x_1 + x_2 + 2x_3 + s = 4 \\ & x_1 + x_2 + x_3 + t = 2 \\ & x_1, x_2, x_3, s, t \geq 0. \end{aligned}$$

To solve the linear program, we are using Dantzig-Wolfe decomposition by keeping the second constraint $x_1 + x_2 + x_3 + t = 2$ in the master problem. That is, we have to consider only the extreme points and rays of the polyhedron $P = \{(x, s) \in \mathbb{R}^3 \times \mathbb{R} \ : \ 2x_1 + x_2 + 2x_3 + s = 4,\ x_1, x_2, x_3, s \geq 0\}$.

(a) Find all extreme points and rays of polyhedron $P$.

(b) Write down the corresponding explicit master problem.

(c) Solve the above linear program using Dantzig-Wolfe decomposition. Use $(x_1, x_2, x_3, s, t) = (0, 0, 0, 4, 2)$ as initial solution.

**Exercise 8** 0.5+0.25+0.3+0.5+0.75 points

Let $n \geq 2$ be an integer. Consider the set

$$X = \Big\{(x, y) \in \{0,1\}^n \times \{0,1\} : y = \prod_{i=1}^{n} x_i\Big\}.$$

(a) Explain why the polyhedron $P$, defined by the inequalities

$$\begin{aligned}
y &\leq x_i, & i &\in \{1, \dots, n\}, & \text{(1a)}\\
\sum_{i=1}^{n} x_i &\leq n - 1 + y, & & & \text{(1b)}\\
x_i &\leq 1, & i &\in \{1, \dots, n\}, & \text{(1c)}\\
-x_i &\leq 0, & i &\in \{1, \dots, n\}, & \text{(1d)}\\
y &\leq 1, & & & \text{(1e)}\\
-y &\leq 0, & & & \text{(1f)}
\end{aligned}$$

is an integer programming formulation of $X$, i.e., $X = P \cap (\mathbb{Z}^n \times \mathbb{Z})$.

(b) Give the definition of the dimension of a polyhedron.

(c) Show that the dimension of $P$ is $n + 1$, in formulae, $\dim(P) = n + 1$.

(d) Prove or refute each of the following statements:

(i) For every $i \in \{1, \dots, n\}$, Inequality (1a) defines a facet of $P$.

(ii) For every $i \in \{1, \dots, n\}$, Inequality (1d) defines a facet of $P$.

(e) Prove that every extreme point of $P$ is an integral vector if $n = 2$.