Examination

Advanced Linear Programming
Spring Semester 2026

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Information

General Information

- The examination lasts 180 minutes.
- The grade is determined as $\max\{1, \text{obtained points}\}$.
- Switch off your mobile phone any other mobile device and put it far away.
- No books or other reading materials are allowed.
- This exam consists of two parts. *Write the answers to the different parts on different pieces of exam paper.* Please write down your name on every exam paper that you hand in.
- Part 1 has 3 questions and part 2 has 3 questions.
- Answers have to be provided in English.
- All your answers should be clearly written down and provide a clear explanation. Unreadable or unclear answers may be judged as false.
- The maximum score per question is given in the first line of the respective question.
- Give your answers to the two parts on separate sheets!!!

GOOD LUCK

Part 1

Recall the *Generalized Steiner Tree* problem from the lecture (aka *Steiner Forest*):

- **Given:**
  - a graph $G = (V, E)$, costs $c : E \to \mathbb{R}_+$,
  - terminal pairs $\{s_i, t_i\}$, for $i \in [n]$.
- **Goal:** Find $F \subseteq E$ containing, for every $i \in [n]$, exactly one s_i-t_i-path, minimizing $\sum_{e \in F} c_e$.

Exercise 1 0.5+0.5+1 points

(a) Formulate Generalized Steiner Tree as an ILP.

(b) Derive the dual of the LP relaxation.

(c) Design a primal-dual approximation algorithm for Generalized Steiner Tree and show that it is a 2-approximation algorithm.
[Hint: You may use without proof the fact that for any tree $T = (V, E)$, the average degree of any set $V' \subseteq V$ that contains all leafs of T is at most 2.]

Exercise 2 2 points

Consider the *Online Generalized Steiner Tree* problem, which is defined identically to the Generalized Steiner Tree problem, except that the terminal pairs arrive sequentially and each pair needs to be connected upon arrival. Upon arrival of a terminal pair, the algorithm can only add edges to F, but not remove any. Design and analyze an online primal-dual algorithm that produces a fractional $O(\log |E|)$-competitive solution to Online Generalized Steiner Tree.

Exercise 3 1 point

Consider again Online Generalized Steiner Tree, but assume that you are also given a forest F' that constitutes a feasible solution (think of it as a predicted optimal solution), and as input a parameter $\lambda \in [0, 1]$. Design an algorithm that produces a solution of cost that is for any input instance simultaneously bounded by above by

- $O\left(\frac{1}{1-\lambda}\right)$ times the cost of F', and
- $O\left(\log\left(\frac{|E|}{\lambda}\right)\right)$ times the cost of an optimal solution for that instance.

You do not need to analyze this algorithm.

Part 2

Exercise 4 1 point

Consider the integer program

$10 + \frac{9}{2} + s_2 = 17.$

$\frac{29}{2} = 17 - \frac{29}{2} = s_2.$

$= \frac{34 - 29}{2} = \frac{5}{2}.$

$$
\begin{aligned}
\max\ & x + 2y \\
& -2x + 5y \leq 15 \\
& 2x + 3y \leq 17 \\
& \boxed{3x + 2y \leq 18} \\
& \boxed{x \leq 5} \\
& x, y \geq 0 \\
& x, y \in \mathbb{Z}
\end{aligned}
$$

$-10 + \frac{15}{2} + s_1 = 15.$

$\frac{15}{2} + s_1 = 25.$

$s_1 = 25 - \frac{15}{2}.$

$= \frac{50 - 15}{2}.$

$= \frac{35}{2}$

The point $(x^\star, y^\star) = (5, \frac{3}{2})$ is an extreme point of its LP relaxation. Derive a Gomory cut for $(x^\star, y^\star)$.

Exercise 5 0.5+0.5+0.5 points

Consider the polyhedron

$$P = \{x \in \mathbb{R}^2 : x_1 + 2x_2 \leq 4,\ 2x_1 + x_2 \leq 4,\ x_1, x_2 \geq 0\}.$$

For each of the following inequalities, decide whether they are split inequalities for P. Justify your answer. In particular, when an inequality is a split inequality, provide a corresponding split and additional arguments for justifying that the inequality can be derived from this split.

(a) $x_1 + x_2 \leq 2$

(b) $x_1 + x_2 \leq 3$

(c) $x_1 + 2x_2 \leq 2$

$x_1 + 2x_2 = 4.$

$2y = 4 - x$

$y = 2 - \frac{1}{2}x.$

$x + y = 2$

$y = 2 - x$

$x_1 + 2y = 2.$

$2y = 2 - x.$

$y = 1 - \frac{1}{2}x.$

Exercise 6 0.5+0.5+0.75+0.75 points

Let $n \geq 2$ be an integer. Consider the set

$$X = \Big\{(x, y) \in \{0,1\}^n \times \{0,1\} : y = \prod_{i=1}^{n} x_i\Big\}.$$

An integer programming formulation of X is given by the inequalities

$$y \leq x_i, \qquad i \in \{1, \dots, n\}, \tag{1a}$$

$$\sum_{i=1}^{n} x_i \leq n - 1 + y, \tag{1b}$$

$$x_i \leq 1, \qquad i \in \{1, \dots, n\}, \tag{1c}$$

$$-x_i \leq 0, \quad = x_i \geq 0. \qquad i \in \{1, \dots, n\}, \tag{1d}$$

$$y \leq 1, \tag{1e}$$

$$-y \leq 0, \tag{1f}$$

which define a polyhedron $P \subseteq \mathbb{R}^{n+1}$, that is, $X = P \cap (\mathbb{Z}^n \times \mathbb{Z})$.

(a) Give the definition of the dimension of a polyhedron.

(b) Show that the dimension of P is $n+1$, in formulae, $\dim(P) = n + 1$.

(c) Prove that Inequality (1b) defines a facet of P.

(d) Prove of refute the following statement: Inequality (1d) defines a facet of P.