Exam "Dynamical Systems" MasterMath

15 January 2019, 10:00-13:00

Answer each of the following questions, making sure that you give compete reasoning for your answers. You are not allowed to use the book, your notes, or a calculator.

1.[15%] Let $X = [0,1) \times [0,1)$ and define $f : X \to X$ by

$$f(x_1, x_2) := (5x_2 \bmod 1, 2x_1 \bmod 1).$$

Is f topologically mixing? Compute the topological entropy of f. *Hint:* consider $f^2 = f \circ f$.

2.[15%] Recall that $\Sigma_m := \{(\ldots, x_{-1}; x_0, x_1, \ldots) \mid x_i \in \{0, \ldots, m-1\}\}$. We define $X \subset \Sigma_m$ by

$$X = \{(\ldots, x_{-1}; x_0, x_1, \ldots) \in \Sigma_m \mid x_{i+1} \neq x_i \text{ for all } i\}$$

Is X a subshift of finite type? If so, what are the forbidden words? Compute the number of allowed words $|W_n(X)|$ of length n (where $n \in \mathbb{N}$). Compute the topological entropy of $\sigma|_X$.

3.[20%] Let $X \subset \mathbb{R}^m$ be a compact set and $f : X \to X$ a Lipschitz continuous map with Lipschitz constant larger than 1. This means that there is a constant $L > 1$ so that

$$||f(x) - f(y)|| \leq L\,||x - y|| \text{ for all } x, y \in \mathbb{R}^m .$$

Prove that $h_{\text{top}}(f) \leq m \log L$ from the definition of topological entropy. *Hint:* use little boxes of width ε/L^n.

4.[20%]

(a) Give an example of a map of the circle that is discontinuous at exactly one point and does not have invariant Borel probability measures. $\frac{1}{7}(x - d)$

(b) Give an example of a continuous map of the real line that does not have invariant Borel probability measures. $f(x) = 1 + x^2$

5.[10%] A diffeomorphism $f : M \to M$ is expansive if there is $\delta > 0$ such that for any two distinct points $x, y \in M$, there is some $n \in \mathbb{Z}$ such that $d(f^n(x), f^n(y)) \geq \delta$. Let Λ be a hyperbolic set of $f : M \to M$. Prove that the restriction of $f|_\Lambda$ is expansive.

4.[20%]

(a) Let Λ_i be a hyperbolic set of $f_i : M_i \to M_i$, $i = 1, 2$. Is $\Lambda_1 \times \Lambda_2$ a hyperbolic set of $f_1 \times f_2 : M_1 \times M_2 \to M_1 \times M_2$?

(b) Let $M = M_1 \times M_2$ be a product of manifolds and let $\pi : M \to M_1$ be the coordinate projection. Suppose that Λ is a hyperbolic set of $f : M \to M$. Let $g : M_1 \to M_1$ be a factor of f with semiconjugacy π. Is $\pi(\Lambda)$ a hyperbolic set of g?