Exam **Spatial Statistics**,
June 27, 2019, 2–5 pm.

This is a closed book exam. Electronic devices connected to the internet are not permitted. Please answer all questions clearly and legibly, and make sure your name and student identification number are on every sheet of paper that you hand in.

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1. We consider an area A, where samples on rainfall are to be collected. A researcher aims to apply model based sampling.

   a Give a brief description of model based sampling, in particular explain the word 'model'.

   The researcher decides on making the best possible map for the area and for that pupose she relies on the kriging variance as the objective function. She can either minimize the average kriging variance and the maximum kriging variance.

   b Explain briefly the difference among the two criteria.

   Taking the minimization of the maximum kriging variance, she decides to first consider three points, with coordinates $(-d/2, 0)$, $(d/2, 0)$ and $(0, d\sqrt{3}/2)$, i.e. on an equilateral triangle with sides equal to d. Within that triangle, the point with the highest kriging variance is the point of gravity. This is the point with coordinates $(0, d\sqrt{3}/6)$. It has distance equal to $d/\sqrt{3}$ to each of the three points. The researcher has found that a linear variogram model without a nugget effect is appropriate for her variable, i.e. $\gamma(h) = |h|$.

   c Give the matrix Γ with the variogram values between the three points, and the vector γ_0 with the variogram values between the three points and prediction point.

   Assuming simple kriging, where the kriging variance equals $\gamma_0'\Gamma^{-1}\gamma_0$. The kriging variance will be a function of d.

   d Determine the kriging variance, i.e. solve the expression $\gamma_0'\Gamma^{-1}\gamma_0$ as a function of d. You may use that

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}^{-1} = 0.5 \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}.$$

   e For which value of d would she have a kriging variance equal to 1?

   f Give at two good arguments why in practical circumstances (say for $d = 10m$) this serves as an approximation only.

[10 points for this question, p.t.o.]

2. **a** With an expression, how is the variogram and covariance related?

   b Given an isotropic exponential variogram having a sill of $s = 20.0$ and range of $a = 100$km and no nugget effect, write the equation of its corresponding covariance.

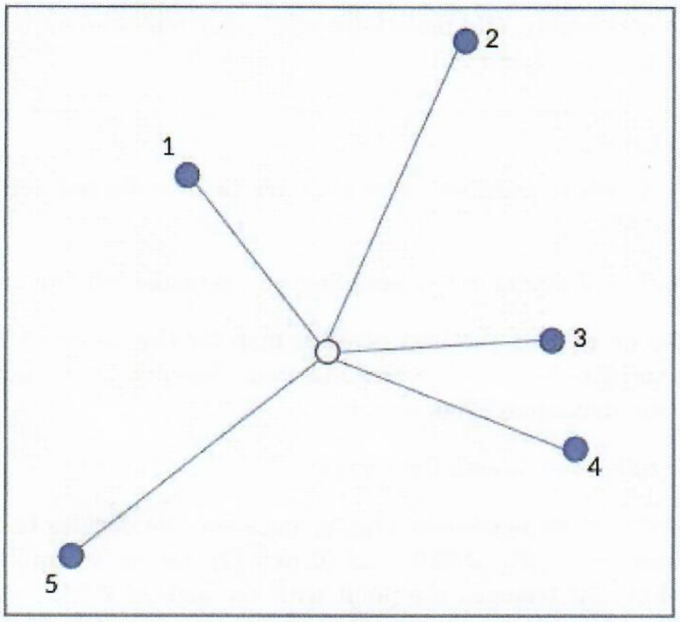

Figure 1: Configuration of five sites.

Consider the configuration of five sites shown in Figure 1. Using the variogram information provided in [**b**] above, the kriging weights λ_i, $i = 1, \ldots, 5$, are estimated for each observation location based on the assumption of known constant mean of $\mu(s) = 160.37$.

c Estimate the value at s_0 using the observed data values

Sites	y
1	122.0
2	183.0
3	148.0
4	160.0
5	176.0

d The kriging weights do not sum up to 1, that is $\sum_i \lambda_i \neq 1$. Briefly comment why?

[10 points for this question]

3. **a** Fill in the sentence with the following phrases:
   intrinsic stationarity, **second-order stationarity**

   Covariance is to ... while semi-variogram is to ...

p.t.o.

Consider the spherical variogram model

$$\gamma(h) = \begin{cases} c_0 + (c - c_0)\left(\left(\frac{3h}{2a}\right) - \frac{1}{2}\left(\frac{h}{a}\right)^2\right) & \text{if } 0 \leq h < a \\ c & \text{if } h \geq a \end{cases}$$

where h is the lag distance in kilometers (km) and a, c_0 and c are the range, nugget, and sill parameters, respectively. If the parameters are estimated as $a = 120$ km, $c_0 = 2.5$, $c = 15$

b What is the semivariance at lag zero?

c What is the semivariance between observations separated by 130 km?

d What is the covariance between observations separated by 125 km?

[5 points for this question]

4. Let $\mathbf{x} \subset [0,1]^2$ be a realisation of a stationary point process observed in the unit square.

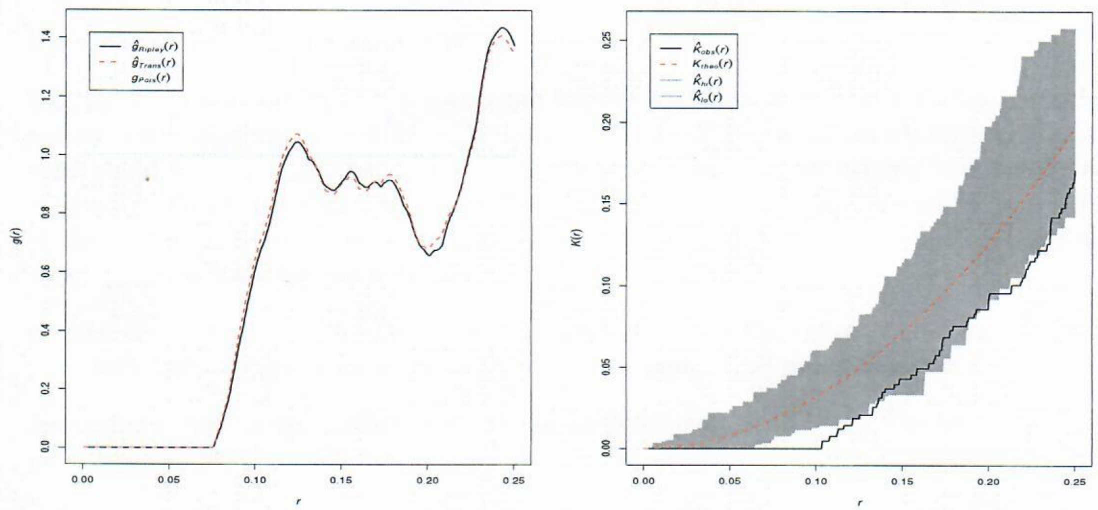

Figure 2: Estimated pair correlation function (left) and K-function (right).

a Give the definitions of the pair correlation function and K-function of a stationary point process.

b The estimated pair correlation function based on $\mathbf{x}$ is plotted in the left-most panel of Figure 2. Give a motivated interpretation.

c Which two properties define *Complete Spatial Randomness* (CSR)?

d To investigate whether $\mathbf{x}$ could be a realisation of a point process that satisfies CSR, consider the K-function and local envelopes based on 99 independent simulations. What is your conclusion based on the plot in the right-most panel of Figure 2?

e Describe a formal two-sided test for CSR based on the K-function at level 0.01.

[10 points for this question, p.t.o.]

5. Let X be a Poisson process on $[0,1]^2$ with intensity function $\lambda(x,y) = \beta e^{\alpha y}$ for $\alpha, \beta \geq 0$ and $(x,y) \in [0,1]^2$.

a What is the log likelihood $L(\alpha, \beta)$? Give an explicit expression.

b Estimate the parameters α and β based on an observed pattern $\mathbf{x} = \{(x_i, y_i) : i = 1, \ldots, 10\}$ in the unit square for which $\sum_i y_i = 5.82$.

c Test the null hypothesis that $\alpha = 0$.

[10 points for this question]