Exam Markov chains, part 2 of module 8 (201400434)
Wednesday June 17, 2020, 8.45 – 10.45 hrs.
Coordinator and teacher: W.R.W. Scheinhardt

This exam consists of **four exercises**.
Please start each exercise on a new sheet of paper.
Put your name and student number on each sheet you submit.
Motivate your answers and use your time efficiently

Integrity statement

Please read the following paragraph carefully, copy the text below it verbatim to the first page of your work (handwritten), add your full name and program (Math/...) and sign it with your signature. If you fail to do so, your test will not be graded.

By testing you remotely in this fashion, we express our trust that you will adhere to the ethical standard of behaviour expected of you. This means that we trust you to answer the questions and perform the assignments in this test to the best of your own ability, without seeking or accepting the help of any source that is not explicitly allowed by the conditions of this test.

The **only allowed sources** for this test are:

- the book by Ross as used in this module (hardcopy or pdf)
- the slides (printed or pdf)
- your own summaries/notes (but no solutions to tutorial/exam problems).
- an ordinary (scientific) calculator (not a programmable or graphic calculator)
- other electronic devices (laptop/tablet/mobile phone), but only to be used:
  - for downloading the test
  - to show the test/book/slides on your screen
  - to write the test (if you prefer to use a tablet instead of paper to write on)
  - for making scans or photos of your work and uploading them to Canvas

Copy this text (handwritten), add your full name and program, and sign it with your signature:

I will make this test to the best of my own ability, without seeking or accepting the help of any source not explicitly allowed by the conditions of the test.

1. Assume that male and female patients apply for medical treatment at a clinic according to two independent Poisson processes with rates (per hour) λ for men and μ for women.

   (a) Determine the probability that the third patient to arrive is a woman.

   (b) Determine the probability that by the time the second woman arrives, at most one man has arrived yet.

   From now on suppose we know that yesterday between 9.00 and 17.00 hrs two men and two women applied for treatment.

   (c) Determine the probability that the first patient to arrive yesterday was a woman.

   (d) Determine the probability that by 15.00 hrs one man and one woman had applied for treatment.

2. Consider a gambler playing a sequence of independent games in which he either wins 1 euro (with probability p) or loses 1 euro. Let X_n be the profit (compared to the initial state) after playing n games. The initial state is $X_0 = 0$, and the gambler quits playing upon reaching a profit of either N or $-N$.

   (a) Why is $\{X_n,\ n = 0, 1, \ldots\}$ a discrete time Markov chain (DTMC)?

   (b) Give the state space, transition matrix, and communicating classes.

   (c) Determine the probability the gambler ends up with a positive profit (of N).

   (d) Give all the stationary (steady-state) distributions of $\{X_n\}$.

   (e) For $N = 2$, determine the expected total number of games after which the gambler has profit 0.

   (f) Again assume $N = 2$, so the gambler quits playing upon reaching a profit of 2 or -2, but in addition the gambler also quits playing with probability $\frac{1}{2}$ upon returning to state 0 (leading to a final profit of 0). In the alternative case upon returning to state 0 (also with probability $\frac{1}{2}$) the gambler continues playing the next game as usual. Determine the probability the gambler will quit with profit 0.

3. Consider a branching process $\{X_n,\ n = 0, 1, \ldots\}$ where each individual has i children with probability p_i, given by $p_0 = \frac{1}{8}, p_1 = \frac{3}{8}$ and $p_2 = \frac{1}{2}$, independent of all else. Assume $X_0 = 1$.

   (a) Determine the expected population size after n generations.

   (b) Determine the probability that the population will eventually die out.

4. Cars arrive at a (very) small unmanned gas station according to a Poisson process with rate λ. There is only room for one car, and arriving cars who find the station occupied leave immediately. Each car has i people in it with probability $p_i = 2^{4-i}/15,\ i = 1, 2, 3, 4$. Since heavy cars use more petrol, filling up the tank of a car with i passengers takes an exponential amount of time with rate $(5 - i)\mu$. After filling up, each car leaves immediately (with all passengers). Let $X(t)$ be the total number of passengers present at time t in the gas station, then $\{X(t), t \geq 0\}$ is a Markov chain in continuous time with state space $\{0, 1, 2, 3, 4\}$.

   (a) Give the generator matrix Q.

   (b) Apply uniformisation to the Markov chain $\{X(t)\}$ using a suitable uniformisation rate, and give the transition matrix P^* of the uniformised DTMC (discrete time Markov chain).

   (c) Let $P_{ij}(t) = P(X(t) = j|X(0) = i)$ and let P^{*n}_{ij} be the (i, j)-th element of $(P^*)^n$. Is $\lim_{t\to\infty} P_{ij}(t) = \lim_{n\to\infty} P^{*n}_{ij}$? Why (not)?

   Now assume $\mu = \lambda$.

   (d) Argue why the stationary distribution of $\{X(t)\}$ does not depend on the value of λ.

   (e) Let $P_{\lambda ij}(t) = P(X(t) = j|X(0) = i)$ for the process $\{X(t)\}$ with parameter λ. Prove, using Kolmogorov equations, that $P_{\lambda ij}(t) = P_{1ij}(\lambda t)$, and interpret (explain) the meaning of this statement.

<u>Norm:</u>

1				2						3		4					Total
a	b	c	d	a	b	c	d	e	f	a	b	a	b	c	d	e	
1	3	2	2	1	3	2	2	3	3	1	2	2	2	2	2	3	36

Grade = Total/4 + 1