Exam Markov chains, part 2 of module 8 (201400434)
Friday July 3, 2020, 13.45 – 16.45 hrs.
Coordinator and teacher: W.R.W. Scheinhardt

This exam consists of **four exercises**.
Please start each exercise on a new sheet of paper.
Put your name and student number on each sheet you submit.
Motivate your answers and use your time efficiently

Integrity statement

Please read the following paragraph carefully, copy the text below it verbatim to the first page of your work (handwritten), add your full name and program (Math/...) and sign it with your signature. If you fail to do so, your test will not be graded.

By testing you remotely in this fashion, we express our trust that you will adhere to the ethical standard of behaviour expected of you. This means that we trust you to answer the questions and perform the assignments in this test to the best of your own ability, without seeking or accepting the help of any source that is not explicitly allowed by the conditions of this test.

The **only allowed sources** for this test are:

- the book by Ross as used in this module (hardcopy or pdf)
- the slides (printed or pdf)
- your own summaries/notes (but no solutions to tutorial/exam problems).
- an ordinary (scientific) calculator (not a programmable or graphic calculator)
- other electronic devices (laptop/tablet/mobile phone), but only to be used:
  - for downloading the test
  - to show the test/book/slides on your screen
  - to write the test (if you prefer to use a tablet instead of paper to write on)
  - for making scans or photos of your work and uploading them to Canvas

Copy this text (handwritten), add your full name and program, and sign it with your signature:

I will make this test to the best of my own ability, without seeking or accepting the help of any source not explicitly allowed by the conditions of the test.

1. Consider large vessels (ships) coming into Rotterdam harbor, according to a Poisson process with rate α per hour. With fixed probability p, independent of the Poisson process, each vessel is an oil tanker.

   (a) Determine the probability that during a day no oil tankers arrive.

   (b) Determine the probability that by the time the second oil tanker arrives, at most one vessel of other type has arrived yet.

   (c) If during some day 2 oil tankers and 1 other vessel arrived, what is the probability that all three arrived in the evening (i.e., after 18.00 hrs.)

   Now suppose the probability p is not fixed, but varies over time as

$$p(t) = \tfrac{1}{2} + \tfrac{1}{2}\cos(\tfrac{t}{24}\tau) \quad \left(= \tfrac{1}{2} + \tfrac{1}{2}\cos(\tfrac{t}{12}\pi)\right),$$

   where $t \in (0, 24)$ is the time of day (in hours), and $\tau = 2\pi \approx 6.2832\ldots$.

   (d) What is the probability of one oil tanker and no other vessels arriving during an evening?

2. Consider a discrete-time Markov chain $\{X_n,\ n = 0, 1, \ldots\}$ with state space $\{1, \ldots, 6\}$ and transition matrix

$$P = \begin{bmatrix} 0 & p & 0 & 0 & 0 & q \\ q & 0 & p & 0 & 0 & 0 \\ 0 & q & 0 & p & 0 & 0 \\ 0 & 0 & q & 0 & p & 0 \\ 0 & 0 & 0 & q & 0 & p \\ p & 0 & 0 & 0 & q & 0 \end{bmatrix}.$$

   Here $p \in [0, 1]$ is a fixed (but unspecified) parameter, and $q = 1 - p$.

   (a) Explain why $\{X_n\}$ is an irreducible, positive recurrent Markov chain for all values of p.

   (b) Determine the (unique) stationary distribution of $\{X_n\}$ (which in fact does exist here).

   (c) Determine the period of state 1 and explain the behaviour of $P^n_{1,1}$ as n grows large for all values of p.

   (d) Starting from state 1, determine the probability that when $\{X_n\}$ enters state 4 for the first time, it does so coming from state 3 (and not coming from state 5).

   (e) Starting from state 1, determine the expected number of visits to state 2, before entering any of the states 3, 4, 5.

3. Consider a branching process $\{X_n,\ n = 0, 1, \ldots\}$ where each individual in each generation has offspring according to the same binomial distribution $Z \sim \mathrm{Bin}(2, \frac{1}{2})$, independent of all else. Assume $X_0 = 1$.

   (a) Determine $E[X_n]$ for $n = 1, 2, \ldots$

   (b) Determine $P(X_2 = 0)$.

   (c) Determine $\lim_{n\to\infty} P(X_n = 0)$.

   (d) Determine $\lim_{n\to\infty} E[X_n | X_n > 0]$.

4. During operation of a machine, minor defects occur according to a Poisson process at rate λ. Whenever the machine has collected i minor defects ($i = 1, 2, \ldots$), it may suffer a major defect in any time interval of length h with probability $\mu_i h + o(h)$, where the $\mu_i,\ i = 1, 2, \ldots$ are nonnegative constants and $o(.)$ is the well-known small-order symbol. In case of a major defect, the machine is immediately replaced by a new machine. Also assume that the machine which starts operating at time 0 is new. Let $X(t)$ be the number of minor defects collected by the current machine (at time t).

   (a) Give the condition(s) needed to conclude that the process $\{X(t), t \geq 0\}$ is a CTMC (continuous time Markov chain).

   Henceforth assume these condition(s) to hold, so that $\{X(t)\}$ is a CTMC.

   (b) Give the generator matrix Q.

   (c) If possible, apply uniformisation to the Markov chain $\{X(t)\}$, using a suitable uniformisation rate, and give the transition matrix P^* of the uniformised DTMC (discrete time Markov chain).
   Do this (only) for the following cases:
   (i) $\mu_i = \mu,\ \ i = 1, 2, \ldots$ for some constant $\mu > 0$;
   (ii) $\mu_i = i\mu,\ i = 1, 2, \ldots$ for some constant $\mu > 0$.

   (d) Write down the Kolmogorov forward equations for $P'_{0j}(t)$), and give the solution for $P_{0j}(t)$ in the particular case where $\mu_i = 0,\ i = 1, 2, \ldots$.

Norm:

1				2					3				4				Total
a	b	c	d	a	b	c	d	e	a	b	c	d	a	b	c	d	
1	3	2	2	2	2	3	2	2	1	2	2	2	2	2	3	3	36

Grade = Total/4 + 1