Exam Markov chains (202001368)
Wednesday June 16, 2021, 9.00 – 11.00 hrs.
(Part 2 of module 8 for Math)
Coordinator and teacher: W.R.W. Scheinhardt

This exam consists of **four exercises**.
Put your name and student number on each sheet you submit.
A standard scientific calculator (not graphic or programmable) is allowed.
Use proper notation and motivate your answers

1. Orders arrive at some company according to a Poisson process with rate λ. Let $N(t)$ be the number of orders arriving between time 0 and t. Each order is an important order with probability p, independently of all else. Let $M(t)$ be the number of important orders arriving between time 0 and t.

   (a) When do you expect the second important order to arrive after time 0?

   (b) Same question, but assuming no orders arrived before time $t = 3$.

   (c) Same question, but now assuming that 3 important orders arrived before time $t = 3$.

   The 'second definition' of the Poisson process states that $N(0) = 0$, that $N(t)$ has stationary and independent increments, and that it gives the form of $P(N(h) = 1)$ and $P(N(h) \geq 2)$ for small values of h, using the 'small order symbol'. Recall that $f(t) = o(t)$ means that $\lim_{t \to 0} \frac{f(t)}{t} = 0$.

   (d) Give the forms for $P(N(h) = 1)$ and $P(N(h) \geq 2)$.

   (e) Use the 'second definition' (see above) to prove that the process $\{M(t)\}$ is a Poisson process.

2. Consider a branching process in which each individual has k children in the next generation with probabilities P_k given by $P_0 = 1/2, P_1 = P_2 = 1/4$. Given that the population starts with a single individual in generation 0, determine:

   (a) The expected population size in generation 3.

   (b) The expected population size in generation 2, given that the population has not died out by then.

3. In a 'bed-and-breakfast' lodging house with two rooms, each day i requests for a room arrive with probability p_i, $i = 0, 1, 2, \ldots$, independently of other days. It is known that $p_0 = p_1 = 1/3$. Requests are accepted when rooms are available. Furthermore, after spending the night the occupant(s) of a room will leave with probability $q = 1/2$, independent of anything else, or otherwise stay for (at least) another day so that in that case the room remains occupied. Let X_n be the number of rooms in use on day n.

(a) Explain why $\{X_n, n = 0, 1, 2, \ldots\}$ is a Markov chain and determine the transition probability matrix P.

(b) Given that today one room is occupied, compute the probability that in each of the next two days at least one room will be occupied.

(c) Show that the vector $\frac{1}{7}(1, 2, 4)$ represents a stationary distribution; is this stationary distribution unique?

(d) If the bed-and-breakfast is empty today, what is the expected number of days before it will be empty again?

4. Two independent and identical machines are either operating or in repair. Operating times and repair times for machine k ($k = 1, 2$) are both exponentially distributed with respective parameters μ and λ. Let $X_k(t) = 1$ if machine k is operating at time t, and $X_k(t) = 0$ otherwise. Then the process $\{(X_1(t), X_2(t))\}$ is a Markov chain in continuous time (CTMC) with four states (i, j), $i = 0, 1$, $j = 0, 1$.

(a) Give the generator matrix Q .

(b) Give the forward Kolmogorov equation for state $(0, 0)$

(c) Is it difficult to give the solution to the system of all four forward Kolmogorov equations in this case? Why (not)? You need not actually (try to) give the solution.

(d) Choose a suitable uniformizing rate v and give the transition matrix P^* of the corresponding uniformized discrete time Markov chain (DTMC).

(e) This DTMC has the same limiting probabilities as $\{(X_1(t), X_2(t))\}$ for any suitable choice of v, but how does the choice of v influence *how fast* the n-step transition probabilities (for the DTMC) converge to the limiting probabilities?

Norm:

1					2		3				4					Total
a	b	c	d	e	a	b	a	b	c	d	a	b	c	d	e	
2	2	2	2	3	2	3	3	2	2	2	2	2	2	3	2	36

Grade = Total/4 + 1