Course	:	Systems & Control (MasterMath)
Date	:	5 December 2022
Time	:	10:00-13:00
Location	:	Science Park Utrecht

Please provide motivation for all your answers and calculations. This is NOT an open book exam, but you are allowed to use a handwritten 'cheat paper' of one page A4, single sided. The use of electronic devices is not allowed nor needed.

1. (Weak solution) Consider the i/o equation

$$\frac{d}{dt}y + y = u, \tag{1}$$

and let $\mathfrak{B}$ be the corresponding behavior.

(a) (2p) For which values of $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ does there exist a pair $(u, y) \in \mathfrak{B}$ with

$$u(t) = \begin{cases} \alpha & t \leq 0 \\ \gamma & t \geq 1 \end{cases} \qquad y(t) = \begin{cases} \beta & t \leq 0 \\ \delta & t \geq 1 \end{cases}$$

(b) (3p) Define (u, y) as:

$$u(t) = \begin{cases} 0 & t \leq 0 \\ 1 + t & 0 \leq t \leq 1 \\ 1 & t \geq 1 \end{cases} \qquad y(t) = \begin{cases} 0 & t \leq 0 \\ t & 0 \leq t \leq 1 \\ 1 & t \geq 1. \end{cases}$$

Show that $(u, y) \in \mathfrak{B}$, that is, (u, y) is a weak solution of (1).

2. (Controller and observer canonical form)
Consider the i/o system described by

$$\frac{d^2}{dt^2}y + 3\frac{d}{dt}y + 2y = (\frac{d}{dt} + 2)u. \tag{2}$$

(a) (1p) Give the controller canonical form of the i/o system (2).

(b) (2p) Eliminate the state from the controller canonical form to obtain an i/o equation.

(c) (1p) How does the newly obtained i/o equation relate to the original i/o equation (2).

(d) (1p) Give the observer canonical form of the original/o system (2).

(e) (2p) Eliminate the state from the ~~controller~~ observer canonical form to obtain an i/o equation.

(f) (1p) How does the newly obtained i/o equation relate to the original i/o equation (2).

(g) (2p) Explain why after elimination of the state from the controller canonical form, you do not obtain the original i/o equation (2), while for the observer canonical form you do get the ordinal i/o equation (2).

3. (Linearisation) Consider the nonlinear system

$$y - \frac{2}{y-1} + 4\frac{d}{dt}y + \frac{d^2}{dt^2}y = 0 \tag{3}$$

(a) (1p) Define $x := [y \; \frac{d}{dt}y]^T$, and determine $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\frac{d}{dt}x = f(x)$, $y = x_1$ is a state space representation of (3).

(b) (2p) Determine the equilibrium points of $\frac{d}{dt}x = f(x)$.

(c) (2p) Linearise the system about each of the equilibrium points.

(d) (3p) Prove that both equilibrium points are locally asymptotically stable.

(e) (2p) Explain why none of the equilibrium points is globally asymptotically stable.

4. (Controllability and pole-placement)
Consider the system $\dot{x} = Ax + Bu$ with

$$A = \begin{bmatrix} 2 & 2 \\ -3 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

(a) (1p) Is the system controllable?

(b) (3p) Find a Kalman controllability decomposition.

(c) (2p) Is the system stabilizable by state feedback? If yes, find a state feedback that results in an asymptotically stable closed loop.

5. (Observability) Consider the behavior given by $R(\frac{\mathrm{d}}{\mathrm{dt}})w = M(\frac{\mathrm{d}}{\mathrm{dt}})\ell$ with

$$R(\xi) = \begin{bmatrix} \xi & \xi^2+4 \\ 1 & \xi-1 \\ 0 & 1 \\ \xi^2+2\xi+2 & \xi^2+1 \end{bmatrix}, \quad M(\xi) = \begin{bmatrix} \xi+3 & \xi^2+3\xi & \xi+2 \\ 0 & 1 & \xi \\ \xi+2 & \xi^2+2\xi-1 & 2 \\ \xi+3 & \xi^2+3\xi & \xi+2 \end{bmatrix}$$

(a) (3p) Is ℓ observable from w?

(b) (2p) Is ℓ detectable from w?

Grade: $1 + \dfrac{9P}{36}$