Mathematical Optimization

Exam April 19, 2022, 9:00 - 12:00

No additional materials may be used during this exam (no notes, calculators, etc.). With this exam a list of theorems and lemmata is provided. In your proofs, you may use definitions from the lecture notes and the theorems and lemmata from the list without providing a proof (reference the theorem/lemma that you use). In addition, you may use all results from Appendix A and all theorems, lemmata, corollaries and propositions from Chapters 6 (Convex Sets), 7 (Convex Functions) and 9 (Iterative Optimization Methods) in the Lecture Notes (v. January 31, 2022) with a reference like "We know that... ".

This exam has 8 exercises.

1. Let $\mathbf{A} = (a_{ij}) \in \mathbb{S}^{n\times n}$ be a positive definite matrix.

   (a) Show that if $\mathbf{M} \in \mathbb{R}^{n\times n}$ is invertible, then $\mathbf{MAM}^T$ is also a positive definite matrix.

   (b) Apply the Diagonalization Algorithm to

   $$\mathbf{A} = \begin{pmatrix} 2 & 8 & 12 \\ 8 & 35 & 49 \\ 12 & 49 & 76 \end{pmatrix}$$

   to find an invertible matrix $\mathbf{Q} \in \mathbb{R}^{3\times 3}$ such that $\mathbf{QAQ}^\mathsf{T}$ is diagonal.

2. Let W be a linear subspace of $\mathbb{R}^n$. Consider the problem of finding the best approximation $\hat{\mathbf{x}} \in W$ of $\mathbf{x} \in \mathbb{R}^n$. That is,

   $$\|\mathbf{x} - \hat{\mathbf{x}}\|_2 = \min_{\mathbf{y}\in W} \|\mathbf{x} - \mathbf{y}\|_2 .$$

   (a) Suppose we found $\mathbf{y} \in W$ such that $\mathbf{x} - \mathbf{y}$ is orthogonal to every $\mathbf{w} \in W$. Prove that $\mathbf{y}$ is the unique best approximation of $\mathbf{x}$ in $\mathbb{R}^n$.

   Let $W^\perp = \{\mathbf{v} \in \mathbb{R}^n \,|\, \mathbf{v}^\mathsf{T}\mathbf{w} = 0, \ \forall \mathbf{w} \in W\}$ be the linear subspace of $\mathbb{R}^n$ consisting of all vectors that are orthogonal to all vectors in W. Denote the best approximation of $\mathbf{x} \in \mathbb{R}^n$ in $W^\perp$ by $\mathbf{z}$.

   (b) Proof that

   $$\mathbf{z} = \mathbf{x} - \mathbf{y} .$$

3. Apply the Fourier-Motzkin procedure to find a solution to the system:

$$\begin{array}{rcrcrcr}
3x & + & y & - & 2z & \leq & 1 \\
 & - & 2y & - & 4z & \leq & -14 \\
-x & + & 3y & - & 2z & \leq & -2 \\
 & & y & + & 4z & \leq & 13 \\
2x & - & 5y & + & z & \leq & 0
\end{array}$$

4. Consider the primal linear program

$$\begin{array}{ll} \max & \mathbf{c}^\mathsf{T}\mathbf{x} \\ \text{s.t.} & \mathbf{A}\mathbf{x} \leq \mathbf{b} \end{array} \tag{P}$$

and its corresponding dual

$$\begin{array}{ll} \min & \mathbf{b}^\mathsf{T}\mathbf{y} \\ \text{s.t.} & \mathbf{A}^\mathsf{T}\mathbf{y} = \mathbf{c} \\ & \mathbf{y} \geq \mathbf{0}, \end{array} \tag{D}$$

where $\mathbf{A} = (a_{ij}) \in \mathbb{R}^{m \times n}$, $\mathbf{b} = (b_i) \in \mathbb{R}^m$, and $\mathbf{c} = (c_j) \in \mathbb{R}^n$. Let $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^m$ be feasible solutions of (P) and (D), respectively, and let $\mathbf{s} = \mathbf{b} - \mathbf{A}\mathbf{x}$.

Prove that $s_i y_i = 0$ for all $i \in \{1, \ldots, m\}$ if and only if $\mathbf{x}$ and $\mathbf{y}$ are optimal solutions of (P) and (D), respectively.

5. (a) Let $f : \mathbb{R}^n \mapsto \mathbb{R}$ be a convex function on $\mathbb{R}^n$ and $\bar{\mathbf{x}}$ a local minimizer.
Prove that $\bar{\mathbf{x}}$ is a global minimizer (without using the theorem that states exactly this fact).
   (b) Prove that the function $f(\mathbf{x}) = \|\mathbf{x}\|$ is convex for any norm $\|.\|$ on $\mathbb{R}^n$.
   (c) Show that $f(x) = e^x$ is convex on $\mathbb{R}$ and derive that $1 + x \leq e^x$ holds for all $x \in \mathbb{R}$.

6. Given the function $f(\mathbf{x}) = \frac{1}{2}x_1^4 - 5x_1^2 + 2x_1x_2 - x_2^2$.
Determine the critical points of f on $\mathbb{R}^n$ and investigate which of these are local minimizer(s).

7. Consider the function $f : \mathbb{R}^n \to \mathbb{R}$ given by

$$f(\mathbf{x}) = \|\mathbf{x}\|_2^2 .$$

(a) Given conjugate directions $d_0, \dots, d_k$ and the point $\mathbf{x}_k$, give $\mathbf{d}_{k+1}$, the direction of the Conjugate Gradient Method for f in iteration $k+1$.

(b) Argue that the Steepest Descent Method with exact line minimization finds the minimizer ($\mathbf{x} = \mathbf{0}$) of f in at most n steps independent of the starting point $\mathbf{x}_0$.

8. Can the Subgradient Method be applied to a function $f : \mathbb{R}^n \to \mathbb{R}$ that is *not* convex? Argue why or why not.

Points: 90 + 10 = 100

1.	(a)	: 7 pt.		5.	(a)	: 7 pt.
	(b)	: 8 pt.			(b)	: 4 pt.
2.	(a)	: 7 pt.			(c)	: 8 pt.
	(b)	: 5 pt.		6.		: 8 pt.
3.		: 10 pt.		7.	(a)	: 5 pt.
4.		: 10 pt.			(b)	: 5 pt.
				8.		: 6 pt.