Mathematical Optimization

Exam July 7, 2021, 9:00 - 12:00

No additional materials may be used during this exam (no notes, calculators, etc.). With this exam a list of theorems and lemmata is provided. In your proofs, you may use definitions from the lecture notes and the theorems and lemmata from the list without providing a proof (reference the theorem/lemma that you use). In addition, you may use all results from Chapter 1 and all theorems, lemmata, corollaries and propositions from Chapter 4 in the Lecture Notes with a reference like "We know that...".

1. Consider the following system of inequalities

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & 3x_3 & \leq & -1 \\ -2x_1 & + & x_2 & & & \leq & 2 \\ & & -5x_2 & - & 6x_3 & \leq & -1 \end{array} .$$

   (a) Apply the Fourier-Motzkin elimination procedure to show the infeasibilty of this system.

   (b) Find a vector $\mathbf{y} \in \mathbb{R}^3$, as mentioned in Farkas Lemma, that exhibits the infeasibilty of this system.

2. Recall that $L(\mathbf{v}_1, \ldots, \mathbf{v}_n) = \{\sum_{j=1}^n \mathbf{v}_j \lambda_j \mid \lambda_j \in \mathbb{Z}\}$ denotes the lattice generated by the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$.

   (a) Compute linearly independent vectors $\mathbf{c}_1$ and $\mathbf{c}_2$, such that

$$L(\mathbf{c}_1, \mathbf{c}_2) = L(\begin{pmatrix} 2 \\ 6 \end{pmatrix}, \begin{pmatrix} 7 \\ 5 \end{pmatrix}, \begin{pmatrix} 12 \\ 12 \end{pmatrix}).$$

   (b) Decide if there is an integer solution $\mathbf{x}$ to the system

$$\begin{pmatrix} 2 & 7 & 12 \\ 6 & 5 & 12 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 2 \\ -10 \end{pmatrix}.$$

   If such $\mathbf{x}$ exists, provide one.

3. Find a matrix $\mathbf{A}$ such that $\mathbf{S} = \mathbf{A}\mathbf{A}^T$ for

$$\mathbf{S} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 7 & 2 \\ 3 & 2 & 15 \end{pmatrix}.$$

4. Consider the primal linear program

$$\begin{array}{ll} \max & \mathbf{c}^T\mathbf{x} \\ s.t. & \mathbf{A}\mathbf{x} \leq \mathbf{b}\,, \end{array}$$

where $\mathbf{A} = (a_{ij}) \in \mathbb{R}^{m\times n}$, $\mathbf{b} = (b_i) \in \mathbb{R}^m$, and $\mathbf{c} = (c_j) \in \mathbb{R}^n$.

(a) Give the dual of this program.

(b) Consider $\mathbf{s} = \mathbf{b} - \mathbf{A}\mathbf{x}$, for a feasible solution $\mathbf{x}$ of the primal problem, and suppose $\mathbf{y} \in \mathbb{R}^m$ is a feasible solution of the dual problem.
Show that if $s_i y_i = 0$ for all $i \in \{1, \ldots, m\}$, then $\mathbf{x}$ and $\mathbf{y}$ are optimal solutions of the primal, respectively, the dual program.

5. Let $f : \mathbb{R}^n \mapsto \mathbb{R}$ be a convex function on $\mathbb{R}^n$. Let M denote the set of (global) minimizers of f on $\mathbb{R}^n$, $M := \{\bar{\mathbf{x}} \in \mathbb{R}^n \mid f(\bar{\mathbf{x}}) \leq f(\mathbf{x}), \forall \mathbf{x} \in \mathbb{R}^n\}$.

Show that the set M is closed and convex.

6. (a) Prove that the function $f(x) = e^x$ is convex.

(b) Use (a) to prove that $e^x \geq ex$ for all $x \in \mathbb{R}$.

7. Let $f(\mathbf{x}) = \frac{1}{2}x_1^4 + 2x_1x_2 + 2x_1 + (1 + x_2)^2$.

(a) Determine the critical points and the local minimizer(s) of f.

(b) Does f have (a) global minimizer(s)? Motivate!

8. We consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(\mathbf{x}) = {x_2}^2(1 + e^{x_1}) + x_1 \sin(x_2) + (x_1 + 2)x_2\,.$$

(a) Show by calculation that

$$\nabla^2 f(-\ln 2, \pi) = \begin{pmatrix} \frac{1}{2}\pi^2 & \pi \\ \pi & 3 \end{pmatrix}.$$

(b) Apply one step of Newton's method to the minimize f starting at $\mathbf{x}_0 = (-\ln 2, \pi)$.

(c) Is the direction that you found in (b) a descent direction? Motivate!

Points: 90 + 10 = 100

1.	(a)	:	6 pt.	4.	(a)	:	2 pt.	7.	(a)	:	10 pt.
	(b)	:	3 pt.		(b)	:	8 pt.		(b)	:	6 pt.
2.	(a)	:	10 pt.	5.		:	7 pt.	8.	(a)	:	2 pt.
	(b)	:	4 pt.	6.	(a)	:	4 pt.		(b)	:	8 pt.
3.		:	10 pt.		(b)	:	6 pt.		(c)	:	4 pt.