Mathematical Optimization

Exam April 13, 2021, 9:00 - 12:00

No additional materials may be used during this exam (no notes, calculators, etc.). With this exam a list of theorems and lemmata is provided. In your proofs, you may use definitions from the lecture notes and the theorems and lemmata from the list without providing a proof (reference the theorem/lemma that you use). In addition, you may use all results from Chapter 1 and all theorems, lemmata, corollaries and propositions from Chapter 4 in the Lecture Notes with a reference like "We know that...".

1. Eliminate x, y, z successively to find a solution to the system

$$\begin{array}{rcrcrcr} 3x & + & y & - & 2z & \leq & 1 \\ & - & 2y & - & 4z & \leq & -14 \\ -x & + & 3y & - & 2z & \leq & -2 \\ & & y & + & 4z & \leq & 13 \\ 2x & - & 5y & + & z & \leq & 0 \end{array}$$

2. For this exercise, you may use that for any two matrices $\mathbf{A}, \mathbf{B}$ of proper dimensions, we have that $\det \mathbf{AB} = \det \mathbf{A} \cdot \det \mathbf{B}$, as well as all standard ways that you know to expand determinants.

   Let $\mathbf{A}$ be a symmetric $n \times n$-matrix. Let $\lambda_1, \ldots, \lambda_n$ denote the eigenvalues of $\mathbf{A}$ (eigenvalues with multiplicity k appear k times).

   (a) Show that $\det \mathbf{A} = \prod_{i=1}^{n} \lambda_i$.

   (b) Use (a) to show that $\det \mathbf{A} > 0$ if $\mathbf{A}$ is positive definite.

   (c) Let $\mathbf{A}_{(i)}$ denote the $i \times i$ north west principle sub-matrix of $\mathbf{A}$ that consists of the intersection of the first i rows and columns of $\mathbf{A}$.

   Show that, if $\mathbf{A}$ is positive definite, then $\det \mathbf{A}_{(i)} > 0$ for all $\mathbf{A}_{(i)}$.

3. Consider the primal linear program

$$\begin{array}{lrcl} \max & \mathbf{c}^T\mathbf{x} & & \\ s.t. & \mathbf{A}\mathbf{x} & \leq & \mathbf{b}\,. \end{array} \qquad (1)$$

where $\mathbf{A} = (a_{ij}) \in \mathbb{R}^{m\times n}$, $\mathbf{b} = (b_i) \in \mathbb{R}^m$, and $\mathbf{c} = (c_j) \in \mathbb{R}^n$.

(a) Give the dual of this program.

(b) Consider $\mathbf{s} = \mathbf{b} - \mathbf{A}\mathbf{x}$, for a feasible solution $\mathbf{x}$ of the primal problem, and suppose $\mathbf{y} \in \mathbb{R}^m$ is a feasible solution of the dual problem.
Show that if $s_i y_i = 0$ for all $i \in \{1,\dots,m\}$, then $\mathbf{x}$ and $\mathbf{y}$ are optimal solutions of the primal, respectively, the dual program.

(c) Now consider the primal linear program

$$\begin{array}{lrcrcl} \max & 5x_1 & + & 3x_2 & & \\ s.t. & 2x_1 & + & x_2 & \leq & 2 \\ & x_1 & - & x_2 & \leq & 2 \\ & 3x_1 & + & 2x_2 & \leq & 4\,. \end{array}$$

Show that $\mathbf{x} = (0,2)^T \in \mathbb{R}^2$ is an optimal solution of the primal problem by computing a dual feasible solution $\mathbf{y} \in \mathbb{R}^3$ using the relation in part (b).

4. (a) Give the definition of a convex function $f : \mathbb{R}^n \to \mathbb{R}$.

(b) Show that the set of global minimizers of a convex function $f : \mathbb{R}^n \to \mathbb{R}$ is a convex set.

Now assume that the convex function $f : \mathbb{R}^n \to \mathbb{R}$ has a unique global minimizer $\overline{\mathbf{x}}$ that is contained in the unit ball, i.e., $\|\overline{\mathbf{x}}\| \leq 1$.

(c) Show that there exists an $\alpha > 0$, such that $f(\mathbf{x}) \geq f(\overline{\mathbf{x}}) + \alpha$ for all $\mathbf{x} \in \mathbb{R}^n$ with $\|\mathbf{x}\| \geq 2$.

5. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(\mathbf{x}) = 100\,(x_2 - x_1^2)^2 + (1 - x_1)^2$.

(a) Compute the gradient vector and the Hessian matrix of f.

(b) Find the critical point(s) of f, and investigate the nature of each critical point.

(c) Is f a convex function? Motivate your answer.

(d) Does f have a global minimizer? Motivate your answer.

6. Apply one step of the steepest descent method to the minimization of
$f(\mathbf{x}) = (2x_1 - x_2)^2 + (x_2 + 1)^2$, starting in $\mathbf{x}_0 = (\frac{5}{2}, 2)^T$.

Points: 90 + 10 = 100

1.		:	12 pt.
2.	(a)	:	6 pt.
	(b)	:	4 pt.
	(c)	:	4 pt.
3.	(a)	:	2 pt.
	(b)	:	8 pt.
	(c)	:	8 pt.
4.	(a)	:	4 pt.
	(b)	:	8 pt.
	(c)	:	8 pt.
5.	(a)	:	5 pt.
	(b)	:	5 pt.
	(c)	:	4 pt.
	(d)	:	4 pt.
6.		:	8 pt.