Script Mathematical Optimization (2021/2022)

Lemma 1 (Taylor's formula). For a C^2-function $f : U \to \mathbb{R}$, with $U \subseteq \mathbb{R}^n$, and $\|\mathbf{d}\|$ small enough:

$$f(\mathbf{x}_0 + \mathbf{d}) = f(\mathbf{x}_0) + \nabla f(\mathbf{x}_0)\mathbf{d} + \frac{1}{2}\mathbf{d}^T \nabla^2 f(\mathbf{x}_0)\mathbf{d} + o(\|\mathbf{d}\|^2)$$

or for, additionally, some $\tau \in (0,1)$:

$$f(\mathbf{x}_0 + \mathbf{d}) = f(\mathbf{x}_0) + \nabla f(\mathbf{x}_0)\mathbf{d} + \frac{1}{2}\mathbf{d}^T \nabla^2 f(\mathbf{x}_0 + \tau\mathbf{d})\mathbf{d}\,.$$

Corollary 2 (LU-factorization). For every matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$, there exists an $(m \times m)$-permutation matrix $\mathbf{P}$ and an invertible lower triangular matrix $\mathbf{M} \in \mathbb{R}^{m\times m}$ such that $\mathbf{U} = \mathbf{MPA}$ is upper triangular.

Corollary 3 (Gale's Theorem). Let $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{b} \in \mathbb{R}^m$ be given.
Then exactly one of the following alternatives is true:

(I) There exists some $\mathbf{x} \in \mathbb{R}^n$ such that $\mathbf{Ax} = \mathbf{b}$.

(II) There exists a vector $\mathbf{y} \in \mathbb{R}^m$ such that $\mathbf{y}^T\mathbf{A} = \mathbf{0}^T$ and $\mathbf{y}^T\mathbf{b} \neq 0$.

Theorem 4 (Integer solutions to linear system of equations). Let $\mathbf{A} \in \mathbb{Z}^{m\times n}$ and $\mathbf{b} \in \mathbb{Z}^m$ be given. Then exactly one of the following statements is true:

(I) There exists some $\mathbf{x} \in \mathbb{Z}^n$ such that $\mathbf{Ax} = \mathbf{b}$.

(II) There exists some $\mathbf{y} \in \mathbb{R}^m$ such that $\mathbf{y}^T\mathbf{A} \in \mathbb{Z}^n$ and $\mathbf{y}^T\mathbf{b} \notin \mathbb{Z}$.

Corollary 5 (Identification of positive (semi-)definite matrices). Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ be a symmetric matrix and $\mathbf{Q} \in \mathbb{R}^{n\times n}$ an invertible matrix such that $\mathbf{D} = \mathbf{QAQ}^T$ is diagonal. Then

(a) $\mathbf{A}$ is *positive semidefinite* if and only if all diagonal elements of $\mathbf{D}$ are non-negative.

(b) $\mathbf{A}$ is *positive definite* if and only if all diagonal elements of $\mathbf{D}$ are strictly positive.

Corollary 6 (Identification of 2×2 positive (semi-)definite matrices). Let $\mathbf{A} \in \mathbb{R}^{2\times 2}$ be a symmetric matrix. Then

(a) $\mathbf{A}$ is *positive semidefinite* if and only if all diagonal elements and the determinant of $\mathbf{A}$ are non-negative.

(b) $\mathbf{A}$ is *positive definite* if and only if all diagonal elements and the determinant of $\mathbf{A}$ are strictly positive.

Theorem 7 (Spectral Theorem for Symmetric Matrices). Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ be a symmetric matrix. Then there exists a matrix $\mathbf{Q} \in \mathbb{R}^{n\times n}$ and eigenvalues $\lambda_1, \dots, \lambda_n$ of $\mathbf{A}$ such that

$$\mathbf{Q}^T\mathbf{Q} = \mathbf{I} \quad \text{and} \quad \mathbf{Q}^T\mathbf{A}\mathbf{Q} = \operatorname{diag}(\lambda_1, \dots, \lambda_n) .$$

Note: the columns of $\mathbf{Q}$ form an orthonormal basis of $\mathbb{R}^n$, consisting of eigenvectors of $\mathbf{A}$.

Theorem 8 (Farkas Lemma). Let $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{b} \in \mathbb{R}^m$ be given.
Then exactly one of the following alternatives is true:

(I) $\mathbf{A}\mathbf{x} \le \mathbf{b}$ is feasible.

(II) There exists a vector $\mathbf{y} \ge \mathbf{0}$ such that $\mathbf{y}^T\mathbf{A} = \mathbf{0}^T$ and $\mathbf{y}^T\mathbf{b} < 0$.

Theorem 9 (Strong Duality). Let $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{b} \in \mathbb{R}^m$, $\mathbf{c} \in \mathbb{R}^n$ and be given, and suppose that either:

(I) there exists some $\mathbf{x} \in \mathbb{R}^n$ such that $\mathbf{A}\mathbf{x} \le \mathbf{b}$

or

(II) there exists some $\mathbf{y} \in \mathbb{R}^m$ such that $\mathbf{A}^T\mathbf{y} = \mathbf{c}$ and $\mathbf{y} \ge \mathbf{0}$.

Then

$$\max\{\mathbf{c}^T\mathbf{x} \mid \mathbf{A}\mathbf{x} \le \mathbf{b}\} = \min\{\mathbf{b}^T\mathbf{y} \mid \mathbf{A}^T\mathbf{y} = \mathbf{c}, \mathbf{y} \ge \mathbf{0}\} .$$

If both (I) and (II) are feasible, then optimal solutions $\mathbf{x}$ of (I) and $\mathbf{y}$ of (II) exist and satisfy $\mathbf{c}^T\mathbf{x} = \mathbf{b}^T\mathbf{y}$.

Lemma 10 (Necessary optimality conditions). Let f be a C^2-function on $\mathbb{R}^n$. Then each local minimizer $\overline{\boldsymbol{x}} \in \mathbb{R}^n$ of f satisfies:

(a) (First order condition) $\nabla f(\overline{\boldsymbol{x}}) = \mathbf{0}^T$

(b) (Second order condition) $\mathbf{d}^T\nabla^2 f(\overline{\boldsymbol{x}})\mathbf{d} \ge 0$ for all $\mathbf{d} \in \mathbb{R}^n$.

Lemma 11 (Sufficient optimality condition). Let f be a C^2-function on $\mathbb{R}^n$ and $\overline{\boldsymbol{x}} \in \mathbb{R}^n$ such that $\nabla f(\overline{\boldsymbol{x}}) = \mathbf{0}^T$ holds. Then $\overline{\boldsymbol{x}}$ is a strict local minimizer of f, provided $\overline{\boldsymbol{x}}$ satisfies

$$\mathbf{d}^T\nabla^2 f(\overline{\boldsymbol{x}})\mathbf{d} > 0 \qquad \text{for all } \mathbf{d} \in \mathbb{R}^n \setminus \{\mathbf{0}\}.$$