Exam: Numerical Mathematics (202200241)

S.M. Glas and M. Schlottbom 27th June 2024, 08.45 – 11.45 2023/2024, Q-2B

Give a suitable explanation of your answers!
The use of electronic devices is *not* allowed. A formula sheet is not handed out.

Question 1.

(i) Compute the pivoted LU-decomposition of the matrix

$$\boldsymbol{A} = \begin{pmatrix} 0 & 3 & -2 \\ 4 & -2 & 1 \\ 2 & -1 & 1 \end{pmatrix}.$$

(ii) Use the pivoted LU-decomposition calculated in (i) to compute the determinant of $\boldsymbol{A}$.

(iii) Use the pivoted LU-decomposition calculated in (i) to solve the linear system $\boldsymbol{Ax} = \boldsymbol{b}$ with $\boldsymbol{b} = (6, -6, 0)^T$.

Question 2. Let the linear system $\boldsymbol{Ax} = \boldsymbol{b}$ be given by

$$\begin{pmatrix} 4 & a \\ a & 1 \end{pmatrix} \boldsymbol{x} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

with $a \in \mathbb{R}$.

(i) Formulate the iteration procedure of the Jacobi method, and give the corresponding iteration matrix.

(ii) For what values $a \in \mathbb{R}$ does the Jacobi method converge?

(iii) Choose $a = 1$ and consider the initial guess $\boldsymbol{x}^{(0)} = (0, 0)^T$. Calculate two steps of the Jacobi method by hand and provide all calculation steps (i.e., stop whenever you calculated $\boldsymbol{x}^{(2)}$).

Question 3.

(i) Compute an interpolating polynomial $\Pi(x)$ through the points (x_i, y_i) with $i = 0, 1, 2$ for $(0, 1)$, $(1, 5)$ and $(2, 3)$ using the Lagrange polynomials $L_i(x)$.

(ii) Show that the interpolating polynomial is unique among all polynomials of degree less than or equal to two.

(iii) Consider the perturbed point $(x_1, \tilde{y}_1)$ with $\tilde{y}_1 = 5 + \delta$, $\delta \in \mathbb{R}$, and denote $\tilde{\Pi}(x)$ the corresponding interpolating through the points (x_0, y_0), $(x_1, \tilde{y}_1)$ and (x_2, y_2). Estimate the maximal error $\Pi(x) - \tilde{\Pi}(x)$ for $x \in [0, 2]$.

Question 4. For the approximation of $\int_{-1}^{1} f(x)dx$ consider the quadrature formula

$$Q(f) = w_0 f(-\frac{1}{\sqrt{3}}) + w_1 f(\frac{1}{\sqrt{3}}).$$

(i) Determine w_0, w_1 such that the degree of exactness is at least one.

(ii) Determine the degree of exactness of $Q(f)$.

Question 5. Let $t_i = ih$ for $h > 0$ and $i \in \mathbb{N}_0$, and let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be continuous and Lipschitz-continuous in the second argument. Consider the numerical method

$$y_{i+1} = y_i + hf\big(t_i + \frac{h}{2}, y_i + \frac{h}{2} f(t_i, y_i)\big), \qquad i \geq 0, \tag{1}$$

for the approximation of the solution $y(t)$ of the initial-value problem $y'(t) = f(t, y(t))$ for $t > 0$ and $y(0) = y_0$.

(i) Show that (1) is a one-step method, i.e., specify the increment function.

(ii) Show that the order of consistency of (1) is $p = 2$. **Hint:** Taylor expansions in $t + h/2$.

(iii) Show that (1) is zero-stable.

(iv) Show that (1) is convergent and give the convergence order.

Question 6. We want to approximate the value of π by approximating the root $x^* = \pi$ of $\sin(x)$. To that end, define $M = [\frac{3\pi}{4}, \frac{5\pi}{4}]$, and consider the fixed-point iteration

$$x_{k+1} = \phi(x_k), \qquad \text{with } \phi(x) = x + \frac{\sin(x)}{2}. \tag{2}$$

(i) Show that $\phi : M \to M$, i.e., $\phi(x) \in M$ for all $x \in M$.

(ii) Show that $\phi : M \to M$ is a contraction.

(iii) Show that the sequence $\{x_k\}$ defined by (2) converges to π.

(iv) Let $\varepsilon > 0$. Give a lower bound for k, such that $|x_k - \pi| < \varepsilon$.

Hint 1: Consider $\phi'(x)$, $x \in M$.

Hint 2: You may use that $\cos(3\pi/4) = \cos(5\pi/4) = \sin(5\pi/4) = -1/\sqrt{2}$ and $\sin(3\pi/4) = 1/\sqrt{2}$.

Exercise	**1.**	**2.**	**3.**	**4.**	**5.**	**6.**	**total**	**Grade**
Points	2+2+2	2+2+1	2+2+2	2+2	1+3+2+1	2+2+2+2	36	(Points + 4)/4