UNIVERSITY OF TWENTE.

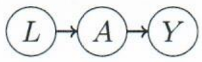
EXAM: Causal Inference

July 2nd, 2024, 13.45-16.45 h.
Instructor: Wouter Koolen

Please answer the questions on separate sheets of paper. Do not forget to number the sheets and to write your name on each of them.

Exercises

1. Consider joint distribution P satisfying the Bayesian network



   (a) [1pt] Give the factorisation of P licensed by the graphical model

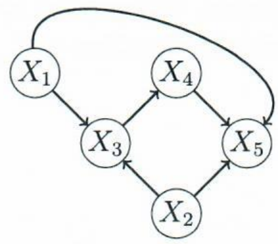
   (b) [1pt] Give the factorisation for the joint distribution $P(L, Y|A = a)$ after conditioning on $A = a$. Is $L \perp\!\!\!\perp Y \mid A = a$?

   (c) [1pt] Give the factorisation for the joint distribution $P(L, Y|do(A = a))$ after intervening by setting $A = a$. Is $L \perp\!\!\!\perp Y \mid do(A = a)$?

2. [3pt] Consider a joint distribution on L, A, Y satisfying (C), (CE) and (P). Suppose that ATE > 0. Is it necessary that $\mathbb{E}[Y^0] \leq \mathbb{E}[Y] \leq \mathbb{E}[Y^1]$? Prove it, or give a counterexample.

3. Consider the Linear Structural Equation Model (left) and corresponding DAG (right)

$$
\begin{aligned}
X_1 &= \epsilon_1 \\
X_2 &= \epsilon_2 \\
X_3 &= \beta_{31}X_1 + \beta_{32}X_2 + \epsilon_3 \\
X_4 &= \beta_{43}X_3 + \epsilon_4 \\
X_5 &= \beta_{51}X_1 + \beta_{54}X_4 + \beta_{52}X_2 + \epsilon_5
\end{aligned}
$$



   where $\epsilon_3, \ldots, \epsilon_5$ are i.i.d. standard normal.

   (a) [2pt] Give the Markov equivalence class and the corresponding CPDAG.

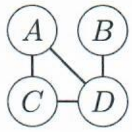
(b) [1pt] Compute the causal effect per unit intervention $\frac{\partial}{\partial x}\mathbb{E}[X_5|do(X_1 = x)]$.

(c) [1pt] Does the backdoor criterion tell us that $\{X_1, X_4\}$ is an adjustment set (aka set of control variables) for the effect of X_3 on X_5?

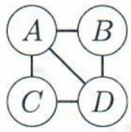
(d) [1pt] Assume that $\beta_{31} \neq 0$. Show that X_1 is an instrumental variable for the effect of X_3 on X_4. Do so by proving that $\mathrm{Cov}(X_4 - \beta_{43}X_3, X_1) = 0$ yet $\mathrm{Cov}(X_3, X_1) \neq 0$.

4. Consider the following two undirected graphical models.

(a)



(b)



(a) [1pt] Does a fully independent joint distribution of A, B, C, D satisfy the constraints expressed by the undirected graphical model (a)?

(b) [1pt] Does any P that satisfies (a) satisfy (b) as well? What about the converse?

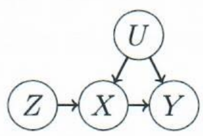
(c) [1pt] Give a conditional independence statement that is true under (b).

(d) [1pt] Give a conditional independence statement that is true under (a) but does not follow from (b).

(e) [1pt] If P is faithful to (a), can it be faithful to (b) as well?

5. [4pt] Consider the NSEM with structure equations and graph

$$
\begin{aligned}
U &= \epsilon_U \\
Z &= \epsilon_Z \\
X &= f(Z) + g(U) + \epsilon_X \\
Y &= \alpha X + j(U) + \epsilon_Y
\end{aligned}
$$



where ϵ_U, ϵ_Z, ϵ_X, and ϵ_Y are independent variables. Assume that we observe Z, X, and Y but not U. Given the distribution rather than a finite sample, regressing X on Z non-parametrically yields the conditional mean $\mathbb{E}[X|Z = z]$ as the regression function. Show that similarly regressing Y on $\mathbb{E}[X|Z]$ (the two-stage least square method) identifies α.

Cheat sheet

Named Assumptions

- Consistency

$$Y = Y^A \tag{C}$$

- Conditional Exchangeability (aka Unconfoundedness)

$$Y^a \perp\!\!\!\perp A \mid L \qquad \text{for all } a \tag{CE}$$

- Positivity

$$\mathbb{P}(A = a|L) > 0 \qquad \text{for all } a \tag{P}$$

Average Treatment Effect $\text{ATE} = \mathbb{E}[Y^1 - Y^0]$.

Faithfulness A joint distribution is *faithful* to an (un)directed graph if for any disjoint sets A, B, S of nodes, $A \perp\!\!\!\perp B \mid S$ if and only if S (d-)separates A from B in the graph.

Open trail A trail between A and B is open given S if

- every interior node in S is a collider
- every interior node not in S is a noncollider or has a descendant in S

d-Separation Sets of nodes A and B are d-separated given S if every trail from A to B is closed.

Backdoor criterion L is an adjustment set if

- L contains only non-descendants of A
- L closes all backdoor trails between A and Y
- $P(A = a|L) > 0$

Markov equivalence Two DAGs are Markov equivalent if and only if they possess the same skeleton and the same set of v-structures.

CPDAG The CPDAG of a Markov equivalence class is the union of its DAGs