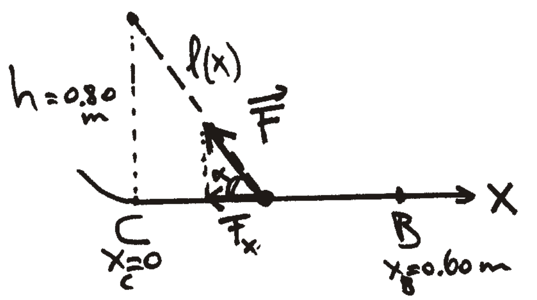
Solutions Dynamica, Exam II, March 19, 2004

1 a. No, the spring force is always perpendicular to the direction of movement in the circular part of the rod

b. Gravity is the only force doing work:

$$\tfrac{1}{2} m v_c^2 = mgh \Rightarrow v_c = \sqrt{2gh} = \sqrt{2gR} = \sqrt{2 \cdot 9.81 \cdot 0.80}$$
$$= 3.96 \text{ m/s}$$

c. The spring is now performing work on the slider



$$\ell(x) = \sqrt{h^2+x^2}$$
$$|\vec{F}| = ku = k(\ell(x) - \ell_0)$$

unstretched length

$$F_x = -|\vec{F}|\cos\alpha = -k(\ell - \ell_0)\frac{x}{\ell} = -kx\left(1 - \frac{\ell_0}{\sqrt{h^2+x^2}}\right)$$

$$W = \int_{x_C}^{x_B} F_x\,dx = -\int_0^{x_B} kx\,dx + \int_0^{x_B} \frac{k\ell_0 x}{\sqrt{h^2+x^2}}\,dx =$$

$$= -\tfrac{1}{2}kx_B^2 + \tfrac{1}{2}\ell_0\int_0^{x_B} k(h^2+x^2)^{-1/2}\,d(h^2+x^2) = -\tfrac{1}{2}kx_B^2 + k\ell_0\sqrt{h^2+x^2}\Big|_0^{x_B} =$$

$$= -\tfrac{1}{2}kx_B^2 + k\ell_0\left(\sqrt{h^2+x_B^2} - \sqrt{h^2}\right) \approx -20\text{ J}$$

$$\tfrac{1}{2}mv_B^2 = \tfrac{1}{2}mv_c^2 + W \Rightarrow v_B = \sqrt{v_c^2 + \frac{2W}{m}} = 1.54 \text{ m/s}$$

d. Then, $\tfrac{1}{2}mv_c^2 + W = 0 \Rightarrow$

$$2mgh - \tfrac{1}{2}kx^2 + k\ell_0\left(\sqrt{x^2+h^2} - h\right) = 0$$

define: $u = \sqrt{x^2+h^2} \Rightarrow u^2 - 2\ell_0 u + C = 0$

with $C = 2\ell_0 h - h^2 - \frac{4mgh}{k} \approx$
$\approx -0.47 \text{ m}^2$

solution: $u = \ell_0\left(1 + \sqrt{1 - C/\ell_0^2}\right) \Rightarrow$

$$\sqrt{x^2+h^2} = \ell_0\left(1+\sqrt{1-C/\ell_0^2}\right) \Rightarrow \sqrt{x^2+h^2} = 1.19\text{ m}$$

$$x = \sqrt{1.19^2 - 0.8^2} = 0.89 \text{ m}$$

2 a. The energy of the compressed spring is converted into kinetic energy of the pea:

$$\frac{1}{2}k\Delta x^2 = \frac{1}{2}mv^2 \Rightarrow v = \sqrt{\frac{k}{m}}\,\Delta x \qquad (1)$$

The time it takes the pea to fall to the ground is independent of its horizontal velocity; therefore: $s = vt$ or:

$$s \sim v \qquad (2)$$

Now: $\frac{s_2}{s_1} = \frac{v_2}{v_1} = \frac{\Delta x_2}{\Delta x_1} \Rightarrow \Delta x_2 = \frac{s_2}{s_1}\Delta x_1 = 0.80\ 2.0\,cm = 1.6\ cm$

b. If $\Delta x_2 = \Delta x_1$ then $v_2 = v_1$.

$$\frac{s_2}{s_1} = \frac{v_1 t_2}{v_1 t_1} = \frac{t_2}{t_1}$$

Since $h = \frac{1}{2}gt^2$ we have $t = \sqrt{\frac{2h}{g}}$, so $\qquad (3)$

$$\frac{t_2}{t_1} = \frac{\sqrt{\frac{2h_2}{g}}}{\sqrt{\frac{2h_1}{g}}} = \sqrt{\frac{h_2}{h_1}} \Rightarrow \frac{h_2}{h_1} = \left(\frac{t_2}{t_1}\right)^2 = \left(\frac{s_2}{s_1}\right)^2 = (0.80)^2 = 0.64$$

$$\Rightarrow h_2 = 0.64 \cdot 0.8\,m = 0.512\ m$$

c. $k = \frac{4.0\,N}{2.0\cdot 10^{-2}\,m} = 200\ N/m$

Combining (1), (2), and (3) we have:

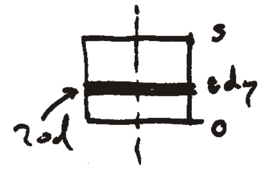
$$s_1 = v_1 \cdot t_1 = \left(\sqrt{\frac{k}{m}}\,\Delta x_1\right)\left(\sqrt{\frac{2h_1}{g}}\right) = \sqrt{\frac{2kh_1}{mg}}\,\Delta x_1 \Rightarrow$$

$$m = \frac{\Delta x_1^2}{s_1^2} \cdot \frac{2kh_1}{g} = \frac{0.020^2}{2.5^2} \cdot \frac{2\cdot 200 \cdot 0.8}{9.81} =$$

$$= 2.1 \cdot 10^{-3}\,kg = 2.1\ g$$

3. a. A thin rod of mass m_r and length L has rotational inertia $I_r = \frac{1}{12} m_r L^2$ with respect to the symmetry axis perpendicular to the rod (see book)
The rotational inertia of a square slab of mass M and size s can be calculated from the above:

$$I_s = \int_0^s dI_{rod} = \int_0^s \frac{1}{12}\underbrace{(dm)}_{=\mu s\,dy} s^2 = \frac{1}{12}\mu s^3 \int_0^s dy =$$



mass per m^2

$$= \frac{1}{12}\mu s^4 = \frac{1}{12} M s^2 \Rightarrow$$

$$I_{large\,slab} = \frac{1}{12} m_s \cdot (2a)^2 = \frac{1}{3} m_s a^2$$

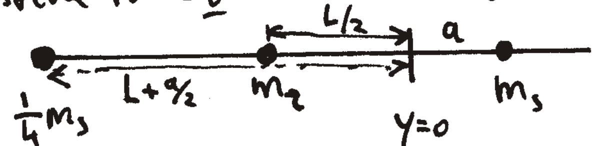
$$I_{small\,slab} = \frac{1}{12} \cdot \left(\frac{1}{4} m_s\right) \cdot a^2 = \frac{1}{48} m_s a^2$$

With the parallel axis theorem, we now can calculate I:

$$I = I_r + m_r \left(\frac{1}{2} L\right)^2 + I_{large\,slab} + m_s \cdot a^2 + I_{small\,slab} + \left(\frac{1}{4} m_s\right)\left(L + \frac{1}{2} a\right)^2 =$$

$$= \frac{1}{3} m_r L^2 + \frac{17}{12} m_s a^2 + \frac{1}{4} m_s L \left(L + \frac{1}{2} a\right)$$

b. The system is equivalent with:



which has the COM:

$$y_{com} = a m_s - \frac{L}{2} m_r - \left(L + \frac{a}{2}\right) \cdot \left(\frac{1}{4} m_s\right) =$$

$$= \left(\frac{7}{8} a - \frac{1}{4} L\right) m_s - \frac{L}{2} m_r$$

$$y_{com} = 0 \Rightarrow m_r = \left(\frac{7}{4} \frac{a}{L} - \frac{1}{2}\right) m_s$$

$$= \frac{3}{8} m_s = 0.9\ \text{kg}$$

3\. c. The total angular momentum before and after this (completely inelastic) collision must be equal:

$$I\omega + L_{bird} = I'\omega'$$

Here $I' = I + m_b R^2$ is the rotational inertia of bird and vane combined, and ω' is the angular velocity after the collision.

$$|\vec{L}_{bird}| = m_b |\vec{r}_b \times \vec{v}_b| = m_b R V, \quad \text{so:}$$

$$\omega' = \frac{I\omega + m_b R V}{I'} = \frac{1.216 \cdot 1.0 + 0.4 \cdot 0.60 \cdot 1.2}{(1.216 + 0.4 \cdot 0.60^2)} =$$

$$= 1.11 \ \text{rad/s}$$

The kinetic energy increase of the wind vane is now:

$$\Delta E = \tfrac{1}{2} I \omega'^2 - \tfrac{1}{2} I \omega^2 = \tfrac{1}{2} \cdot 1.216 (1.11^2 - 1.0^2) = 0.136 \text{ J}$$

d. Angular momentum conservation:

$$I\omega - m_b a V = I\omega' - m_b a V' \Rightarrow$$

$$I(\omega' - \omega) = -m_b a (V - V') \qquad (1)$$

kin. energy conservation:

$$\tfrac{1}{2} I \omega^2 + \tfrac{1}{2} m_b V^2 = \tfrac{1}{2} I \omega'^2 + \tfrac{1}{2} m_b V'^2 \Rightarrow$$

$$I(\omega'^2 - \omega^2) = m_b (V^2 - V'^2)$$

$$I(\omega' - \omega)(\omega' + \omega) = m_b (V - V')(V' + V) \overset{(1)}{\Rightarrow}$$

$$-\cancel{m_b} a \cancel{(V - V')} (\omega' + \omega) = \cancel{m_b} \cancel{(V - V')} (V' + V)$$

$$-a(\omega' + \omega) = V' + V \qquad (2)$$

(1) + $(-m_b a)$ (2) gives:

$$\omega' = \frac{(I - m_b a^2)\omega - 2 m_b a V}{I + m_b a^2} =$$

$$= \frac{(1.216 - 0.064) \cdot 1.0 - 2 \cdot 0.4 \cdot 0.4 \cdot 1.2}{1.216 + 0.064} = 0.6 \ \text{rad/s}.$$

Kinetic energy loss:

$$\Delta E = \tfrac{1}{2} I \omega'^2 - \tfrac{1}{2} I \omega^2 = \tfrac{1}{2} \cdot 1.216 (0.6^2 - 1.0^2) = -0.389 \text{ J}.$$

4. a. Rotational inertia of cylinder of length L and radius R around its vertical symmetry axis:

$$I = \int_{\varphi=0}^{2\pi}\int_{z=0}^{L}\int_{r=0}^{R} \rho\, r^2\, r\,dr\,d\varphi\,dz = 2\pi\rho L\cdot\tfrac{1}{4}R^4 = \tfrac{1}{2}\pi\rho L R^4$$

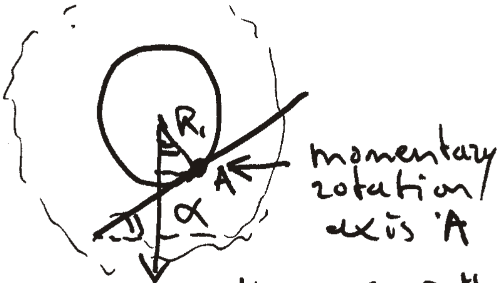
where ρ is the density. For the dumbbell:

$$I_{tot} = \tfrac{1}{2}\pi\rho\, l R_1^4 + 2\cdot\tfrac{1}{2}\pi\rho\left(\tfrac{1}{2}l\right)R_2^4 = \tfrac{1}{2}\pi\rho\, l\left(R_1^4 + R_2^4\right)$$

$$M = \rho\cdot\left(\pi R_1^2 l + 2\cdot\pi R_2^2\left(\tfrac{1}{2}l\right)\right) = \rho\pi l\left(R_1^2 + R_2^2\right) \Rightarrow$$

$$I_{tot} = \frac{1}{2}M\frac{R_1^4 + R_2^4}{R_1^2 + R_2^2}$$

b. around A: angular acceleration



$$I'_{tot}\,\alpha_{cm} = \tau$$

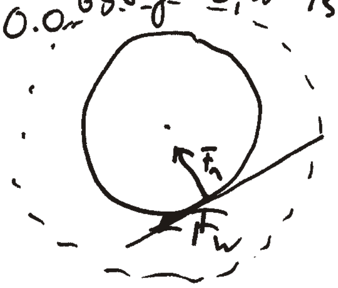
$$\tau = |\vec{\tau}| = |\vec{r}\times\vec{F}_z| = MgR_1\sin\alpha$$

$$I'_{tot} = I_{tot} + MR_1^2 = \frac{1}{2}M\frac{R_1^4+R_2^4}{R_1^2+R_2^2} + MR_1^2 = \frac{1}{2}M\frac{3R_1^4+2R_1^2R_2^2+R_2^4}{R_1^2+R_2^2}$$

(parallel axis theorem)

$$a_{cm} = R_1\alpha_{cm} = \frac{MgR_1^2\sin\alpha}{I'_{tot}} = 2g\sin\alpha\frac{R_1^4+R_1^2R_2^2}{3R_1^4+2R_1^2R_2^2+R_2^4} = 0.0658\,g = 0.65\ \mathrm{m/s^2}$$

c. Around central axis: (same rot. acceleration!)



$$I_{tot}\alpha_{cm} = F_W\cdot R_1 \Rightarrow F_W = \frac{I_{tot}\,a_{cm}}{R_1^2} =$$

$$= \frac{1}{\cancel{2}}M\frac{R_1^4+R_2^4}{\cancel{R_1^2+R_2^2}}\cdot\cancel{2}g\sin\alpha\frac{\cancel{R_1^2+R_2^2}}{3R_1^4+2R_1^2R_2^2+R_2^4} =$$

$$= Mg\sin\alpha\frac{R_1^4+R_2^4}{3R_1^4+2R_1^2R_2^2+R_2^4} \qquad F_n = Mg\cos\alpha \qquad F_W = \mu F_n \Rightarrow$$

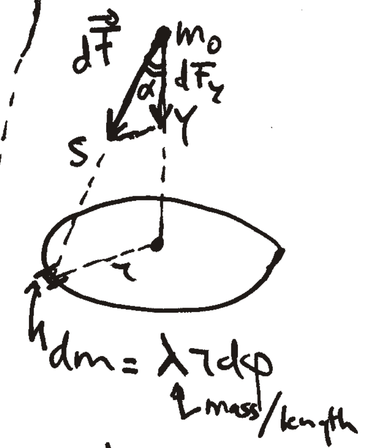
$$\mu_{min} = \frac{F_W}{F_n} = \tan\alpha\frac{R_1^4+R_2^4}{3R_1^4+2R_1^2R_2^2+R_2^4} = 0.50$$

d. $$s = \tfrac{1}{2}a_{cm}t^2 \Rightarrow t = \sqrt{\frac{2s}{a_{cm}}} = \sqrt{\frac{1.0}{0.65}} = 1.24\ \mathrm{s}.$$

5. a. $|d\vec{F}| = G m_0 \frac{dm}{s^2}$; $|d\vec{F}_y| = |d\vec{F}| \cos\alpha = \frac{|d\vec{F}| y}{s}$

$$|\vec{F}| = \int_{\varphi=0}^{2\pi} |d\vec{F}_y| = \frac{G m_0 \lambda y r}{\left(\sqrt{y^2+r^2}\right)^3} \int_0^{2\pi} d\varphi =$$

$$= \frac{G m_0 y (\lambda 2\pi r)}{(y^2+r^2)^{3/2}} = \frac{G m m_0 y}{(y^2+r^2)^{3/2}}$$



$2\pi r \lambda = m$

b. $$\frac{1}{2} m_0 v^2 = -\int_{y=y_0}^{0} |\vec{F}| dy = \int_0^{y_0} \frac{G m m_0 y\, dy}{(y^2+r^2)^{3/2}} =$$

$$= \frac{1}{2} G m m_0 \int_0^{y_0} (y^2+r^2)^{-3/2} d(y^2+r^2)$$

$$= \frac{1}{2} G m m_0 \left[-2 (y^2+r^2)^{-1/2}\right]_0^{y_0} =$$

$$= G m m_0 \left(\frac{1}{r} - \frac{1}{\sqrt{y_0^2+r^2}}\right) \Rightarrow$$

$$v = \sqrt{2Gm\left(\frac{1}{r} - \frac{1}{\sqrt{r^2+y_0^2}}\right)} \qquad (\cdot v = v(0))$$

c. $$v = \sqrt{\frac{2Gm}{r}\left(1 - \frac{1}{\sqrt{1+\left(\frac{y_0}{r}\right)^2}}\right)} \approx \sqrt{\frac{2Gm}{r}} \sqrt{1 - \frac{1}{1+\frac{1}{2}\left(\frac{y_0}{r}\right)^2}}$$

$$\approx \sqrt{\frac{2Gm}{r}} \sqrt{1 - \left(1 - \frac{1}{2}\left(\frac{y_0}{r}\right)^2 + O\left(\left(\frac{y_0}{r}\right)^4\right)\right)} =$$

$$= \sqrt{\frac{2Gm}{r}} \sqrt{\frac{1}{2}\left(\frac{y_0}{r}\right)^2 + O\left(\frac{y_0}{r}\right)^4} = \sqrt{Gm}\, \frac{y_0}{r^{3/2}} + O\left(\frac{y_0}{r}\right)^2$$

d. for $\frac{y}{r} \ll 1$ we can also do a Taylor-expansion of the force acting on m_0:

$$F = \frac{G m m_0 y}{r^3 \left(1+\frac{y^2}{r^2}\right)^{3/2}} \approx \frac{G m m_0}{r^2} \cdot \frac{y}{r} \cdot \left(1 - \frac{3}{2}\frac{y^2}{r^2}\right) \approx$$

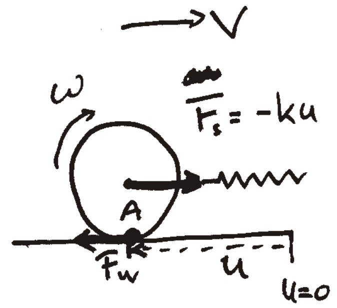
$$\approx \frac{G m m_0}{r^3} y \quad \text{in leading order.}$$

Note that the force is directed towards $y=0$ always, such that

$$m_0 \ddot{y} = -\frac{G m m_0}{r^3} y \Rightarrow \ddot{y} + \frac{Gm}{r^3} y = 0$$

This is a harmonic oscillator with frequency:

$$\omega = \sqrt{\frac{Gm}{r^3}} \Rightarrow f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{Gm}{r^3}}$$



6. At the release position, the system only has potential energy: $\frac{1}{2}ku_{max}^2$
As it passes through the equilibrium position it has only kinetic energy, both translational and rotational: $\frac{1}{2}I\omega^2 + \frac{1}{2}MV^2$

Now: $I = \frac{1}{2}MR^2$ (see book)

$V = \omega R$ (no slipping), so energy balance gives:

$$\frac{1}{2}ku_{max}^2 = \frac{1}{2}I\omega^2 + \frac{1}{2}MV^2 = \left(\frac{1}{2}\frac{I}{R^2} + \frac{1}{2}M\right)V^2 = \frac{1}{2}\left(\frac{1}{2}M + M\right)V^2 \Rightarrow$$

$V = \sqrt{\frac{2k}{3M}}\,u_{max}$ is the velocity at the equilibrium position

a. Translational kinetic energy at equilibrium position:

$$K_{trans} = \frac{1}{2}MV^2 = \frac{1}{\cancel{2}}\cancel{M}\frac{\cancel{2}k}{3\cancel{M}}u_{max}^2 = \frac{1}{3}ku_{max}^2 = 7.5\ J$$

b. Rotational kinetic energy at equilibrium position:

$$K_{rot} = \frac{1}{2}I\omega^2 = \frac{1}{2}\frac{I}{R^2}V^2 = \frac{1}{4}M\frac{2k}{3M}u_{max}^2 = \frac{1}{6}ku_{max}^2 = 3.75\ J$$

(Note that $T_{trans} + T_{rot} = \frac{1}{2}ku_{max}^2$, as it should be.)

c. Since F_w is unknown, calculate the rotational acceleration around point A:

$$\left.\begin{array}{l} I'\alpha = F_s R \\ \ddot{u} = a = \alpha R \text{ (rolling without slipping)} \end{array}\right\} \Rightarrow$$

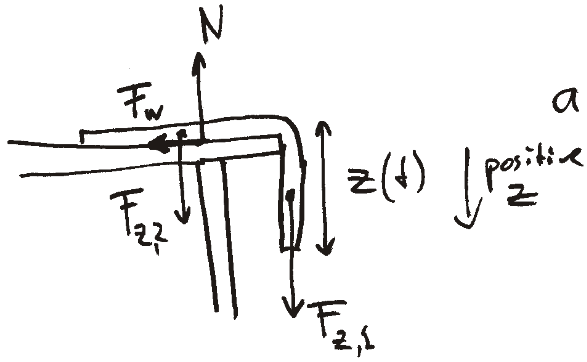
$I' = I + MR^2 = \frac{3}{2}MR^2$ (parallel axis theorem)

$$\frac{3}{2}MR^2 \cdot \frac{\ddot{u}}{R} = -kuR \Rightarrow \frac{3}{2}M\ddot{u} = -ku \Rightarrow$$

$\ddot{u} + \frac{2k}{3M}u = 0$ harmonic oscillator with $\omega^2 = \frac{2k}{3M}$

so: $$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\frac{2k}{3M}}} = 2\pi\sqrt{\frac{3M}{2k}}$$

7.



a. $F_{z,1} = \frac{z}{L} Mg$

$F_{z,2} = \frac{L-z}{L} Mg = N$

$F_w = \mu N = \mu \frac{L-z}{L} Mg$

$$M\ddot{z} = \sum F = F_{z,1} - F_w = \frac{z}{L} Mg - \mu \frac{L-z}{L} Mg =$$

$$= (1+\mu) \frac{Mg}{L} z - \mu Mg \Rightarrow$$

$$\ddot{z} - (1+\mu) \frac{g}{L} z + \mu g = 0 \qquad (1)$$

b. We show that (1) is solved by inserting the function and finding ω, C, A and B that solve the problem:

$$z(t) = Ae^{\omega t} + Be^{-\omega t} + C \Rightarrow$$

$$\dot{z}(t) = \omega A e^{\omega t} - \omega B e^{-\omega t} \Rightarrow$$

$$\ddot{z}(t) = \omega^2 A e^{\omega t} + \omega^2 B e^{-\omega t} = \omega^2 (z(t) - C)$$

Inserting in (1):

$$\omega^2 (z(t) - C) - (1+\mu) \frac{g}{L} z(t) + \mu g = 0 \Rightarrow$$

$$\left(\omega^2 - (1+\mu)\frac{g}{L}\right) z(t) + \mu g - \omega^2 C = 0$$ This holds for all t if:

$$\omega = \sqrt{(1+\mu)\frac{g}{L}} \quad \text{and} \quad C = \frac{\mu g}{\omega^2} = \frac{\mu g}{(1+\mu) g/L} = \frac{\mu L}{1+\mu}$$

From the initial conditions, $z(0) = L/3$, $\dot{z}(0) = 0$, we determine A and B:

$$\dot{z}(0) = 0 \Rightarrow \omega A - \omega B = 0 \Rightarrow A = B$$

$$z(0) = L/3 \Rightarrow A + B + C = L/3 \Rightarrow 2A + \frac{\mu L}{1+\mu} = \frac{L}{3} \Rightarrow$$

$$A = \frac{L}{6} \frac{(1-2\mu)}{1+\mu}$$

So: $z(t) = \frac{L}{6}\left(\frac{1-2\mu}{1+\mu}\right)(e^{\omega t} + e^{-\omega t}) + L\left(\frac{\mu}{1+\mu}\right) =$

$\omega = \sqrt{(1+\mu)\, g/L}$

$\frac{1}{2}(e^{\omega t} + e^{-\omega t}) = \cosh(\omega t)$

$$= \frac{L}{3}\left(\frac{1-2\mu}{1+\mu}\right) \cosh\left(\sqrt{(1+\mu)\frac{g}{L}}\, t\right) + L\left(\frac{\mu}{1+\mu}\right)$$