Exam Mathematical Optimisation (201500379)

Tuesday 18th April, 8.45 – 11.45

Motivate all your answers!

A copy of the lecture-sheets may be used during the examination. You may use any results from the lecture slides in your answers (Lemmas, Theorems, Corollaries, Exercises, etc.), however you should reference the result.

Some hints are provided at the end of this paper.

Good Luck!

1. Let $A \in \mathbb{R}^{n\times n}$ be symmetric. Show that
   - (a) $A + \lambda I \succeq 0$ for λ large enough. [$\succeq 0$ means *p.s.d.*] [2 points]
   - (b) $\min \{x^T Ax |\ x^T x = 1\} = \max \{\lambda | A - \lambda I \succeq 0\}$. [3 points]
   - (c) if $A = uu^T + vv^T$ for some $u, v \in \mathbb{R}^n$ then $rank(A) \leq 2$. [2 points]

2. Let $A = [a_1, \ldots, a_n] \in \mathbb{Z}^{m\times n}$ be a matrix whose colomns span R^m.
   - (a) Give the definition of lattice basis for the lattice $L(a_1, \ldots, a_n)$. [1 point]
   - (b) Suppose $C = [a_{i_1}, \ldots, a_{i_m}]$ is a submatrix of A that minimizes $|det\ C|$. Can we conclude that the columns of C form a lattice basis? (Motivate your answer!) [2 points]
   - (c) Suppose $C = [c_1, \ldots, c_m]$ is a matrix with columns in $L(a_1, \ldots, a_n)$ that minimizes $|det\ C|$. Can we conclude that the columns of C form a lattice basis? [2 points]

3. Consider the linear optimisation problem:

$$\begin{aligned} \max_{\mathbf{x}} \quad & 5x_1 + x_2 \\ \text{s.t.} \quad & 3x_1 - x_2 \leq 4 \\ & x_2 \leq 6 \\ & x_1 + x_2 \leq 8 \end{aligned}$$

   - (a) Determine the dual problem to this problem. [2 points]
   - (b) Given that $(3, 5)$ is an optimal solution to the primal problem, find an optimal solution to the dual problem. [3 points]

4. Consider $\emptyset \neq X \subseteq \mathbb{R}^n$ and $f_i : X \to \mathbb{R}$ for $i = 1, \ldots, m$ (neither X nor f_i necessarily convex). Show that the following function $g : \mathbb{R}^m \to \mathbb{R} \cup \{\infty\}$ is convex: [3 points]

$$g(\mathbf{y}) = \max_{\mathbf{x}} \left\{ \sum_{i=1}^{m} y_i f_i(\mathbf{x}) : \mathbf{x} \in X \right\}.$$

P.T.O.

5. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(\mathbf{x}) = (x_1^2 - 5x_1 + x_2)^2 + x_1^2 + x_2^2$, with gradient vector

$$\nabla f(\mathbf{x}) = 2\begin{pmatrix}(2x_1 - 5)x_2 + 2x_1^3 - 15x_1^2 + 26x_1 \\ x_1^2 - 5x_1 + 2x_2\end{pmatrix}.$$

(a) Critical points:

i. Determine the critical points and local minimiser(s) of f on $\mathbb{R}^n$. [3 points]

ii. Is f a convex function? [2 points]

iii. Does f have a global minimiser, and if so where? [2 points]

iv. Does f have a global maximiser, and if so where? [2 points]

(b) Steepest descent and conjugate gradient:

For $\mathbf{x}_k, \mathbf{d}_k \in \mathbb{R}^n$, we let $t_k = \arg\min_t\{f(\mathbf{x}_0 + t\mathbf{d}_0)\}$ and $\mathbf{x}_{k+1} = \mathbf{x}_k + t_k\mathbf{d}_k$.

Given that $\mathbf{x}_4 = (1,\, 2)^T$ and $\mathbf{g}_3 = (2,\, 2)^T$ and $\mathbf{d}_3 = (0,\, -1)^T$:

i. Determine the direction of steepest descent of f at $\mathbf{x}_4$? [1 point]

ii. Using the Fletcher-Reeves formula for α_4, determine the conjugate gradient direction at $\mathbf{x}_4$. [2 points]

(The directions do not need to be normalised.)

6. For the Quasi-Newton Method, show that

(a) $d_k = -H_k g_k$ is a descent direction, provided $H_k \succeq 0$. [1 point]

(b) Explain why at the next iteration point x_{k+1} we have $g_{k+1}^T d_k = 0$. [3 points]

7. (Automatic additional points) [4 points]

Question:	1	2	3	4	5	6	7	Total
Points:	7	5	5	3	12	4	4	40

Hints:

1. $\begin{pmatrix} a & b \\ b & c \end{pmatrix} \succeq O \quad \Leftrightarrow \quad a \geq 0 \wedge c \geq 0 \wedge b^2 \leq ac$

2. $\begin{pmatrix} a & b \\ b & c \end{pmatrix} \succ O \quad \Leftrightarrow \quad a > 0 \wedge c > 0 \wedge b^2 < ac$