Mathematical Optimization

(Sample Exam 2020/2021)

No additional materials may be used during this exam (no notes, calculators, etc.). With this exam a list of theorems and lemmata is provided. In your proofs, you may use definitions from the lecture note and the theorems and lemmata from the list without providing a proof. In addition, you may use all results from Chapter 1 in the Lecture Notes.

1. Suppose $\mathbf{A} \in \mathbb{R}^{n\times n}$.

(a) Show: if $\mathbf{A}$ is positive definite, then $\mathbf{A}^{-1}$ exists and $\mathbf{A}^{-1}$ is positive definite.

(b) Assume that $\mathbf{A}$ is positive <u>semi</u>definite and that for $\mathbf{x} \in \mathbb{R}^n$ the relation $\mathbf{x}^T\mathbf{A}\mathbf{x} = 0$ is satisfied. Show that then $\mathbf{A}\mathbf{x} = \mathbf{0}$ holds.

2. Consider the system of inequalities

$$\begin{array}{rcl} x_1 + x_2 & \leq & 1 \\ x_1 - x_2 & \leq & -2 \\ -x_2 & \leq & -4 \\ -x_1 + x_2 & \leq & 5 \end{array} \qquad (*)$$

(a) Show using Fourier-Motzkin elimination that the system (*) does not have a solution.

(b) Prove the infeasibility of the system (*) by using alternative (II) of the Farkas Lemma.

3. Consider the linear program

$$\begin{array}{lrcl} \max & \mathbf{c}^T\mathbf{x} & & \\ s.t. & \mathbf{A}\mathbf{x} & \leq & \mathbf{b}\,, \end{array} \qquad (**)$$

where $\mathbf{A} = (a_{ij}) \in \mathbb{R}^{m\times n}$, $\mathbf{b} = (b_i) \in \mathbb{R}^m$, and $\mathbf{c} = (c_j) \in \mathbb{R}^n$.

(a) Give the linear program that is dual to $(**)$.

(b) Suppose that $\bar{\mathbf{x}}$ is an optimal solution to $(**)$ and $\bar{\mathbf{y}}$ is an optimal solution the dual problem. Show (from scratch) that $\mathbf{c}^T\bar{\mathbf{x}} \leq \mathbf{b}^T\bar{\mathbf{y}}$.

(c) Prove Farkas Lemma from the strong duality theorem.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a convex function with $f(0) = 0$.

(a) Show that the functie φ, defined by

$$\varphi(t) = \frac{f(t)}{t}$$

is monotonically increasing for $t > 0$.

(b) Now assume that f is differentiable in 0. Show that from part (a) it follows that for $y > 0$

$$f(y) \geq f'(0)y.$$

5. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = \frac{1}{3}x^3 + 2xy + x + y^2$. We want to investigate the situation concerning the existence of a global minimizer.

(a) Show that $\bar{\mathbf{x}} = (1, -1)$ is the only critical point of f and find the Hessian at $\bar{\mathbf{x}}$. Can you conclude from the Hessian that $\bar{\mathbf{x}}$ is a local minimizer or a local maximizer?

(b) Calculate $f(1 + t, -1 - t)$, for all $t \in \mathbb{R}$. Is $\bar{\mathbf{x}}$ is a local minimizer?

(c) Starting from $\mathbf{x}_0 = (0, 0)$ we want to apply the steepest descent method. Find the descent direction, and investigate the minimization problem. What can you conclude from this?

6. Let be given the quadratic function $q(x)$ on $\mathbb{R}^n$ by

$$q(x) = \frac{1}{2}x^T Ax + b^T x,$$

where A is a positive definite $n \times n$-matrix.

(a) Show that the point $\overline{x} := -A^{-1}b$ is the unique global minimizer of q.

(b) Show that Newton's method finds this global minimizer in just one step, independent of the starting point $\mathbf{x}_0$.

(c) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = \frac{1}{3}x^3 + 2xy + x + y^2$. Apply one step of Newton's method to f, starting from $\mathbf{x}_0 = (2, 0)^T$.