Examination: Mathematical Programming I (158025)

June 28, 2004, 13.30-16.30

Ex.1 Prove the following statements.

(a) Let $\mathbf{A} = (a_{ij}) \in \mathbb{R}^{n \times n}$ be a positive definite matrix. Then $\mathbf{A}$ is regular and also the inverse $\mathbf{A}^{-1}$ is positive definite.

(b) Suppose $\mathbf{A} \geq 0$ and $\mathbf{x}^T \mathbf{A} \mathbf{x} = 0$. Then $\mathbf{A}\mathbf{x} = \mathbf{0}$.

Ex.2

(a) Show that the following system does <u>not</u> have any feasible solution.

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & 3x_3 & \leq & -1 \\ -2x_1 & + & x_2 & & & \leq & 2 \\ & & -5x_2 & - & 6x_3 & \leq & -1 \end{array} .$$

(b) Consider the pair of primal and dual linear programs,

$$\begin{array}{llll} \mathrm{P}: & \max\limits_{\mathbf{x} \in \mathbb{R}^n} \ \mathbf{c}^T \mathbf{x} & \text{s.t.} & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ \mathrm{D}: & \min\limits_{\mathbf{y} \in \mathbb{R}^m} \ \mathbf{b}^T \mathbf{y} & \text{s.t.} & \mathbf{A}^T \mathbf{y} = \mathbf{c} \ , \quad \mathbf{y} \geq \mathbf{0} \ , \end{array}$$

where $\mathbf{A}$ is an $(m \times n)$-matrix $(m \geq n)$ and $\mathbf{c} \in \mathbb{R}^n$, $\mathbf{b} \in \mathbb{R}^m$. Let v_P denote the maximum value of the primal program P and v_D the minimum value of the dual problem D. The feasible sets of P and D are abbreviated by F_P and F_D. Suppose the feasible set F_P is not empty $(F_P \neq \emptyset)$. Show that then we have

$$F_D = \emptyset \quad \text{if and only if} \quad v_P = \infty \ .$$

Ex.3

(a) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function and consider the affine transformation $\mathbf{x} = \mathbf{A}\mathbf{y} + \mathbf{b}$ with $\mathbf{A}$ a $(n \times m)$-matrix and $\mathbf{b} \in \mathbb{R}^n$. Show that the composition $g : \mathbb{R}^m \to \mathbb{R}$, $g(\mathbf{y}) := f(\mathbf{A}\mathbf{y} + \mathbf{b})$ is convex on $\mathbb{R}^m$.

(b) Let be given convex functions $f_j : \mathbb{R}^n \to \mathbb{R}$, $j \in J$, with a finite index set J. Define the function $f(\mathbf{x}) := \max\{f_j(\mathbf{x}) \mid j \in J\}$, $\mathbf{x} \in \mathbb{R}^n$. Show that the function f is convex on $\mathbb{R}^n$.

Ex.4 Given the function $f(\mathbf{x}) = x_1^4 + x_2^4 - 4x_1x_2 + 2$.

(a) Determine the critical points and the local minimizer(s) of f.

(b) Does there exist a global minimizer (on $\mathbb{R}^n$).

Ex.5 Consider the quadratic function $q : \mathbb{R}^n \to \mathbb{R}$,

$$q(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{C}\mathbf{x} + \mathbf{b}^T\mathbf{x} ,$$

with a symmetric matrix $\mathbf{C} \in \mathbb{R}^{n\times n}$ and $\mathbf{b} \in \mathbb{R}^n$. Suppose the function q is bounded from below, *i.e.*, there exists a constant $K \in \mathbb{R}$ such that

$$q(\mathbf{x}) \geq K \quad \text{for all} \quad \mathbf{x} \in \mathbb{R}^n .$$

Under these assumptions:

(a) Show that $\mathbf{C}$ must be positive semidefinite.

(b) For any $\mathbf{x}$ satisfying $\mathbf{C}\mathbf{x} = \mathbf{0}$ it follows $\mathbf{b}^T\mathbf{x} = 0$.

(c) Show: There exists a solution of $\mathbf{C}\mathbf{x} = -\mathbf{b}$. Furthermore, the quadratic function q has a global minimizer on $\mathbb{R}^n$.

Points: 36+4 =40

Ex. 1	a	:	3 pt.
	b	:	3 pt.
Ex. 2	a	:	3 pt.
	b	:	4 pt.
Ex. 3	a	:	3 pt.
	b	:	3 pt.
Ex. 4	a	:	4 pt.
	b	:	3 pt.
Ex. 5	a	:	3 pt.
	b	:	2 pt.
	c	:	5 pt.

The scipt 'Mathematical Programming I' may be used during the examination. Good luck!