Examination: Mathematical Programming I (191580250)

July 6, 2012, 8.45 -11.45

Ex.1 Prove the following statements.

(a) Let $\mathbf{A} = (a_{ij}) \in \mathbb{R}^{n\times n}$ be a positive definite matrix. Show that the matrix $\mathbf{A}$ is nonsingular.

(b) Consider a matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ of the form $\mathbf{A} = \sum_{i=1}^{k} \mathbf{b}_i\mathbf{b}_i^T$ with vectors $\mathbf{b}_i \in \mathbb{R}^n, i = 1,\dots,k$. Show that $\mathbf{A}$ is positive semidefinite.
Under the additional assumption that the vectors $\mathbf{b}_i$ are linearly independent, prove that the rank of $\mathbf{A}$ is k (i.e., $\ker \mathbf{A} = \{\mathbf{x} \mid \mathbf{A}\mathbf{x} = \mathbf{0}\}$ has dimension $n - k$.)

Ex.2 Let be given $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{b} \in \mathbb{R}^m$. Show that precisely one of the following alternatives I or II is true:

$$(I) \qquad \mathbf{A}\mathbf{x} < \mathbf{b} \quad \text{has a solution } \mathbf{x}$$

$$(II) \qquad \mathbf{A}^T\mathbf{y} = \mathbf{0},\ \mathbf{b}^T\mathbf{y} \le 0, \mathbf{y} \ge \mathbf{0}, \mathbf{y} \ne \mathbf{0} \quad \text{has a solution } \mathbf{y}$$

Hint: You may first prove that (I) and (II) cannot hold simultaneously. To finish the proof, use a trick as in the proof of Gordan's corollary

Ex. 3 Let $f : \mathbb{R} \to \mathbb{R}$ be a convex function on $\mathbb{R}$. Suppose there are points $x_1 < x_2 < x_3$ and $a, b \in \mathbb{R}$ such that $f(x_i) = ax_i + b, i = 1, 2, 3$, *i.e.*, the points $(x_i, f(x_i))$ are on a line. Prove that then we must have: $f(x) = ax + b$ for all $x \in [x_1, x_3]$.

Ex. 4

(a) Let $C_i \subset \mathbb{R}^n, i \in I$, be convex sets, where I is some (possibly infinite) index set. Show that the set $C := \bigcap_{i\in I} C_i$ is also convex.

(b) Let $f_i : \mathbb{R}^n \to \mathbb{R}, i \in I$, be convex functions on $\mathbb{R}^n$, where I is some index set. Show that the function $f(x) := \max_{i\in I} \{f_i(x)\}$ is also convex.
Is there a relation between the result in (a) and (b)?

Ex. 5 We wish to compute the minimizer $\overline{\mathbf{x}}$ of the quadratic function $q(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{A}\mathbf{x} + \mathbf{b}^T\mathbf{x}$ with positive definite matrix $\mathbf{A}$.

(a) Determine the minimizer $\overline{\mathbf{x}}$ of q and show that for any starting point $\mathbf{x}_0$ the Newton method finds this minimizer of q in one step.

(b) Let us apply the Quasi-Newton method. Suppose that this method produces the iterates x_k, the search directions $\mathbf{d}_k$ and the matrices $\mathbf{H}_k$, $k = 0, 1, \dots$. Show that the relation holds:

$$\mathbf{H}_k^{-1}d_j = \mathbf{A}d_j \text{ , for all } j = 0, \dots, k-1 ,$$

and after n steps we have $\mathbf{H}_n = A^{-1}$.
Hint. Use the relation (from the proof of Lemma 5.6): $\mathbf{H}_k\gamma_j = \boldsymbol{\delta}_j$, $0 \le j \le k-1$, *where* $\gamma_j = \mathbf{g}_{j+1} - \mathbf{g}_j$, $\boldsymbol{\delta}_j = \mathbf{x}_{j+1} - \mathbf{x}_j$.

Ex. 6 Show: Given $\mu > 0$ and $\mathbf{p} = (p_1, \ldots, p_n)^T \in \mathbb{R}^n$ with $\mathbf{p} > \mathbf{0}$, the function

$$f(\mathbf{x}) := \mathbf{p}^T \mathbf{x} - \mu \sum_{i=1}^{n} \ln(x_i)$$

is convex on $\{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} > \mathbf{0}\}$ and the point $\overline{x} = (\mu/p_1, \ldots, \mu/p_n)^T$ is the unique minimizer of f.

Points: 36+4 =40

Ex. 1	a	:	2 pt.	Ex. 4	a	:	2 pt.
	b	:	4 pt.		b	:	4 pt.
Ex. 2		:	6 pt.	Ex. 5	a	:	3 pt.
Ex. 3		:	5 pt.		b	:	4 pt.
				Ex. 6		:	6 pt.

A copy of the lecture-sheets may be used during the examination. (The copies may not contain worked out exercises.)

Good luck!