Examination: Mathematical Programming I (191580250)

July 2, 2013, 8.45 -11.45

Ex.1

(a) Show that $A \in \mathbb{R}^{n \times n}$ is positive definite if and only if A^{-1} is positive definite.

(b) Let $A \in \mathbb{R}^{n \times n}$ be positive semidefinite and assume that for $x \in \mathbb{R}^n$ the relation $x^T A x = 0$ is satisfied. Show that then $Ax = 0$ holds.

Ex. 2 Consider the system of inequalities

$$\begin{array}{rcl} x_1 + x_2 & \leq & 1 \\ x_1 - x_2 & \leq & -2 \\ -x_2 & \leq & -4 \\ -x_1 + x_2 & \leq & 5 \end{array} \qquad (*)$$

(a) Show with the help of the Fourier-Motzkin elimination that the system (*) is not feasible (does not have a solution).

(b) Prove the infeasibility of the system (*) by using the (dual) alternative (II) of the Farkas Lemma (Theorem2.6).

Ex. 3

(a) Let $f : \mathbb{R}^n \to \mathbb{R}$, be a C^1-function. Show that f is convex if and only if the following inequality holds:

$$\left(\nabla f(x) - \nabla f(x')\right)^T (x - x') \geq 0 \quad \text{for all } x, x' \in \mathbb{R}^n .$$

(Hint: For "$\Leftarrow$" use the mean-value relation:
$f(x) - f(x') = \nabla f(x' + \lambda(x - x'))^T (x - x')$ *for some* $\lambda \in (0, 1)$.
Also recall that $\nabla f(x)$ *is a column vector.*

(b) Use part (a) to prove that the quadratic function $q(x) := \frac{1}{2} x^T A x$ (with symmetric $n \times n$-matrix A) is convex if and only if A is positive semidefinite.

Ex. 4

(a) Let $g : \mathbb{R}^n \to I$, $I \subset \mathbb{R}$ be convex and $f : I \to \mathbb{R}$ be convex and non-decreasing. Show that the composition $f \circ g(\mathbf{x}) = f(g(\mathbf{x}))$ of the functions f and g is convex on $\mathbb{R}^n$.

(b) Show : The function $f(x) = e^{\|\mathbf{x}\|}$ is convex on $\mathbb{R}^n$ (for any norm $\|\mathbf{x}\|$ on $\mathbb{R}^n$).

Ex. 5 Given the function $f(\mathbf{x}) = \frac{1}{2}x_1^4 + 2x_1x_2 + 2x_1 + (1 + x_2)^2$

(a) Determine the critical points and the local minimizer(s) of f.

(b) Show that the local minimiser(s) are global minimizer(s) of f (on $\mathbb{R}^n$).

Ex. 6 Let be given a quadratic function $q(x) = \frac{1}{2}x^T Ax + b^T x$ with positive definite $n \times n$-matrix A.

(a) Show that the point $\overline{x} := -A^{-1}b$ is the unique global minimizer of q (on $\mathbb{R}^n$).

(b) Starting with $x_0 \in \mathbb{R}^n$, let us apply the conjugate gradient method to solve $\min_{x\in\mathbb{R}^n} q(x)$ (as formulated in Theorem 5.3, with descent directions $d_0, \ldots, d_k$).
Show that the iteration point x_{k+1} is the (global) minimizer of the quadratic function $q(x)$ on the affine subspace

$$S_k = \{x = x_0 + \gamma_0 d_0 + .. + \gamma_k d_k \mid \gamma_0, .., \gamma_k \in \mathbb{R}\}$$

Points: 36+4 + 3 (extra points) =43

Ex. 1	a	:	3 pt.	Ex. 4	a	:	4 pt.
	b	:	3 pt.		b	:	2 pt.
Ex. 2	a	:	3 pt.	Ex. 5	a	:	4 pt.
	b	:	3 pt.		b	:	3 **extra points**.
Ex. 3	a	:	4 pt.	Ex. 6	a	:	3 pt.
	b	:	3 pt.		b	:	4 pt.

'tot' = 36

A copy of the lecture-sheets may be used during the examination.
(The copies may not contain worked out exercises.)

$g = \frac{p+4}{4}$

Good luck!

nodig : p=20