Exam Optimal Control code 156162

Date	:	27-01-2005
Location	:	WA 212
Time	:	9.00-12.00

Please provide clear motivation for all your answers and indicate which theorems you are using.
Do not spend too much time on a single item. If you are not able to solve part(s) of a problem, then move on and use those parts as if you have already solved them.

There are four exercises.

1. Consider the following minimization problem:

$$J = \int_0^4 x(t) + \dot{x}(t)^2 dt, \quad x(0) = 0, \quad x(4) = 1$$

(a) Find the (unique) candidate solution of the minimization problem and calculate the corresponding minimal value of J.

In what follows we have to satisfy the additional requirement that $\dot{x}(t) \geq 0$ and $x(t) \geq 0$.

(b) Does your solution of 1a satisfy the constraints?

(c) A pragmatic attitude would be to cut off the solution of 1a at the initial interval, that is, take $x(t) = 0$ for those ts for which the solution $x(t)$ in 1a is negative. Calculate the corresponding value of J.

(d) Define $u = \dot{x}$ and write the problem as an optimal control problem with a constraint on u and x.

(e) Give the Hamiltonian for the optimal control problem and write the candidate optimal control as a function of the co-state p. Make sure that u satisfies the additional constraint.

(f) What can be said about the constraint on x if the constraint on u is satisfied?

(g) Determine the general solution for p. Argue that $p(t)$ changes sign at most once. Call the time instant at which $p(t)$ changes sign t_1.

(h) Determine the optimal control, the corresponding state trajectory and the optimal value of J.

(i) Comment on the three different values of J corresponding to the unconstrained problem, the pragmatic solution to the constrained problem, and the optimal solution to the constrained problem respectively. Explain how they are ordered.

2. Consider the nonlinear system

$$\begin{aligned} \frac{d}{dt}x_1(t) &= x_2(t) \\ \frac{d}{dt}x_2(t) &= -x_1(t) - (\alpha + x_1(t)^2)x_2(t) \end{aligned} \tag{1}$$

Here $\alpha \in \mathbb{R}$ is a parameter.

(a) Determine the equilibrium points of (1).

(b) Linearize (1) about $(0,0)$.

(c) Investigate, using the linearization, the asymptotic stability of $(0,0)$ for both $\alpha > 0$ and $\alpha < 0$.

(d) What can be concluded from the linearization about the (asymptotic) stability for $\alpha = 0$?

(e) Prove that $(0,0)$ is stable for $\alpha = 0$.

3. Consider the system

$$\frac{d}{dt}x = 2x + u$$

The cost criterion is as follows:

$$J = \int_0^1 x(t)^2 + u(t)^2 dt + \int_1^2 2x(t)^2 + u(t)^2 dt$$

The aim is to find a control u that minimizes J. The value function is denoted by $V(t,x)$.

(a) What is nonstandard about this optimal control problem.

(b) Assume that $x(1) = x_1$ is given. Provide the equation(s) through which $V(1, x_1)$ may be calculated. You don't have to *solve* the differential equations.

(c) Explain that the optimization problem on $[0,1]$ is:
Minimize

$$V(1, x(1)) + \int_0^1 x(t)^2 + u(t)^2 dt$$

Give the equations that determine the above optimal control problem. You don't have to *solve* the differential equations.

(d) Combine the previous parts to describe the complete solution to the problem. You don't have to *solve* the differential equations.

4. Consider the system and cost criterion

$$\frac{d}{dt}x = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u \qquad J = \int_0^\infty x(t)^T \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}}_{Q} x(t) + u(t)^2 dt$$

The aim is to minimize J. The initial state is $x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

(a) Determine the optimal control $u = Fx$.

(b) Determine the optimal costs.

(c) Calculate $M := A + BF$. Is M a Hurwitz matrix?

(d) Find a quadratic Lyapunov function for $\frac{d}{dt}x = Mx$.

Grading:

1									2					3				4			
a	b	c	d	e	f	g	h	i	a	b	c	d	e	a	b	c	d	a	b	c	d
4	2	4	3	3	2	3	4	2	4	4	6	3	4	4	6	4	6	6	5	5	6

Grade: $1 + \frac{\text{points}}{10}$