Optimal Control
(course code: 156162)

Date: 22-06-2010
Place: 13:45-16:45
Time: CU B103

1. Consider the nonlinear system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\sin(x_1)(1+x_2) \\ 4x_1 - \sin(2x_2) \end{bmatrix}. \tag{1}$$

   (a) Determine all points of equilibrium.
   (b) Determine the linearization at $\bar{x} = (0, 0)$.
   (c) Let $A \in \mathbb{R}^{2\times 2}$ be the matrix of the linearization at $\bar{x} = (0, 0)$.
      - What are the eigenvalues of A?
      - Find *all diagonal* positive definite 2×2 matrices P such that

$$A^T P + PA = -Q$$

      with Q positive definite.
   (d) Determine a Lyapunov function for the nonlinear system (1) at $\bar{x} = (0, 0)$.

2. What is the difference between global asymptotic stability and global attractivity?

3. Consider the cost

$$J := \int_0^1 (x(t) + \dot{x}(t))^2 \, dt$$

   (a) Find the function $x(t)$ that satisfies the Euler equation for this cost J, with initial and final condition

$$x(0) = 1, \qquad x(1) = \beta.$$

   (Here β is a fixed real number.)
   (b) Does this $x(t)$ satisfy a second order condition of minimality of J?
   (c) Is there a β for which the cost function J is equal to zero? Explain.
   (d) Derive the function $x(t)$ that satisfies the Euler-Lagrange equation and free end-point condition for optimality of J subject to the initial condition $x(0) = 1$ and free end point. (You must use the theory of Euler-Lagrange and free end-point. Any other method is considered invalid.)

4. In control there is a trade-off between how aggressive we are allowed to control the system and how fast the state should converge to zero. With infinite horizon LQ we can analyze this trade-off. To this end let γ be a positive parameter and consider the cost

$$J_{\infty,\gamma} := \int_0^\infty x^2(t) + \gamma u^2(t) \, dt, \qquad \text{with } x(0) = 1 \text{ and } u(t) \in \mathbb{R}$$

   and assume that the state dynamics are

$$\dot{x}(t) = -x(t) + u(t).$$

   Notice that $u(t)$ is not restricted in any sense: every real value $u(t)$ is allowed.

(a) Suppose γ is very large and that $u(t)$ minimizes $J_{\infty,\gamma}$. Considering the formula of $J_{\infty,\gamma}$ do you expect that the $u(t)$ will be "large" or "small"?

(b) Write down the Algebraic Riccati Equation (ARE)

(c) Determine all solutions P of the ARE

(d) Determine all *positive* solutions P of the ARE

(e) Determine the optimal cost $J_{\infty,\gamma}$

(f) The lecture notes asserts that $u(t) = -\frac{1}{\gamma}Px(t)$ is the optimal solution for the positive solution $P > 0$ of the ARE.

- Determine optimal $x(t)$ and $u(t)$ for all $t \geq 0$
- Determine the two limits

$$\lim_{\gamma\to\infty} u(t), \qquad \lim_{\gamma\to\infty} J_{\infty,\gamma}$$

of the optimal $u(t)$ and minimal cost $J_{\infty,\gamma}$.

- Are the limits in agreement with the answer of part (a) of this problem? Specifically explain in words why the limits are not a surprise. (This problem can also be answered if you didn't manage to determine the limits.)

(g) Now suppose we solve instead the *finite* horizon problem

$$\min_u \underbrace{\int_0^{t_e} x^2(t) + \gamma u^2(t)\,dt}_{\tilde{J}_\gamma}$$

subject to the same dynamics $\dot{x} = -x + u$, the same initial initial condition $x(0) = 1$ and zero final condition $x(t_e) = 0$. Argue that for every $t_e > 0$ the optimal cost $\tilde{J}_\gamma$ can never be less than the optimal cost $J_{\infty,\gamma}$ of the infinite horizon problem.

5. Formulate Pontryagin's minimum principle for the case that we optimize over the input u as well as the final time t_e.

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problem:	1	2	3	4	5
points:	3+3+(2+3)+2	3	4+2+1+4	1+2+2+1+2+(2+2+2)+3	4

Exam grade is $1 + 9p/p_{\max}$.

Euler:

$$\left(\frac{\partial}{\partial x} - \frac{d}{dt}\frac{\partial}{\partial \dot{x}}\right) F(t, x(t), \dot{x}(t)) = 0$$

Beltrami:

$$F - \frac{\partial F}{\partial \dot{x}}\dot{x} = C$$

Hamiltonian equations (with $H = p^T f(x,u) + L(x,u)$) for initial conditioned state:

$$\dot{x} = \frac{\partial H}{\partial p}(x, p, u), \qquad x(0) = x_0,$$

$$\dot{p} = -\frac{\partial H}{\partial x}(x, p, u), \qquad p(t_e) = \frac{\partial S}{\partial x}(x(t_e))$$

LQ Riccati differential equation:

$$\dot{P}(t) = -P(t)A - A^T P(t) + P(t)BR^{-1}B^T P(t) - Q, \qquad P(t_e) = G$$

Bellman:

$$\frac{\partial W}{\partial t}(x,t) + \min_{v\in\mathcal{U}}\left[\frac{\partial W}{\partial x^T}(x,t) f(x,v) + L(x,v)\right] = 0, \qquad W(x, t_e) = S(x)$$