Optimal Control
(course code: 156162)

Date: 26-06-2012
Place: HB-2E
Time: 13:45–16:45

1. Consider the nonlinear system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_1^2 + x_2 \\ x_1 - x_2 - 2 \end{bmatrix} \tag{1}$$

   (a) Determine all points of equilibrium
   (b) Determine the linearization at all points of equilibrium
   (c) Does the *nonlinear* system (1) have an unstable equilibrium?
   (d) Does the *nonlinear* system (1) have a stable equilibrium?

2. Consider minimization of

$$\int_0^\pi \dot{x}(t)^2 + \cos(x(t))\,\mathrm{d}t, \qquad x(0) = 0, \quad x(\pi) = 1$$

   over all functions $x : [0,\pi] \to \mathbb{R}$.

   (a) Determine the Euler equation for this problem
   (b) Solve the Euler equation with $x(0) = 0, x(\pi) = 1$
   (c) Are the second order conditions of Legendre satisfied?

3. Consider the second order system with mixed initial and final conditions

$$\dot{x}_1(t) = u(t), \quad \dot{x}_2(t) = 1, \qquad x_1(0) = 0,\ x_2(0) = 0,\ x_1(1) = 2$$

   and with cost

$$J(u) := \int_0^1 u^2(t) + 4x_2(t)u(t)\,\mathrm{d}t.$$

   The input $u : [0,1] \to \mathbb{R}$ is not restricted, i.e. $u(t)$ can take on any real value.

   (a) Determine the Hamiltonian for this problem
   (b) Determine the differential equations for state x and co-state p, including the boundary conditions
   (c) Express the candidate minimizing $u_*(t)$ as a function of $x_*(t), p_*(t)$
   (d) Solve the equations for x_*, p_*, u_* (that is, determine $x_*(t), p_*(t), u_*(t)$ as explicit functions of time $t \in [0,1]$)

4. What is the definition of *value function*
5. Consider the system

$$\dot{x}(t) = x(t) + u(t), \qquad x(0) = x_0, \qquad u(t) \in \mathbb{R}$$

on the infinite time horizon with cost

$$J_\infty(x_0, u) = \int_0^\infty \gamma^2 x^2(t) + u^2(t) \, \mathrm{d}t.$$

Here γ is some nonzero real number.

(a) Determine all solutions P of the Algebraic Riccati Equation

(b) Determine the value function

(c) Find the optimal $u(t), x(t)$ explicitly as a function of time

(d) Explicitly determine the optimal solution $u(t), x(t)$ and value function for the case that $\gamma = 0$. Also, which general result tells you that the optimal solution might be qualitatively different for $\gamma = 0$ compared to $\gamma \neq 0$?

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problem:	1	2	3	4	5
points:	2+2+1+1	2+3+1	1+2+2+3	3	1+2+2+3

Exam grade is $1 + 9p/p_{\max}$.

Euler:

$$\left(\frac{\partial}{\partial x} - \frac{d}{dt}\frac{\partial}{\partial \dot{x}}\right) F(t, x(t), \dot{x}(t)) = 0$$

Beltrami:

$$F - \frac{\partial F}{\partial \dot{x}} \dot{x} = C$$

Standard Hamiltonian equations for initial conditioned state:

$$\dot{x} = \frac{\partial H}{\partial p}(x, p, u), \qquad x(0) = x_0,$$
$$\dot{p} = -\frac{\partial H}{\partial x}(x, p, u), \qquad p(T) = \frac{\partial S}{\partial x}(x(T))$$

LQ Riccati differential equation:

$$\dot{P}(t) = -P(t)A - A^T P(t) + P(t)BR^{-1}B^T P(t) - Q, \qquad P(T) = G$$

Bellman:

$$\frac{\partial W(x,t)}{\partial t} + \min_{v \in \mathscr{U}} \left[\frac{\partial W(x,t)}{\partial x^T} f(x, v) + L(x, v) \right] = 0, \qquad W(x, T) = S(x)$$