Optimal Control
(course code: 156162)

Date: 24-01-2013
Place: Citadel
Time: 13:45–16:45

1. Consider the nonlinear system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_1^2 + x_2 \\ x_1 - x_2 - 2 \end{bmatrix} \tag{1}$$

   (a) Determine all points of equilibrium

   (b) Determine the linearization at all points of equilibrium

   (c) Does the *nonlinear* system (1) have an unstable equilibrium?

   (d) Does the *nonlinear* system (1) have a stable equilibrium?

2. Consider minimization of

$$\int_0^T \dot{x}(t)^2 + 4t\dot{x}(t)\,\mathrm{d}t, \qquad x(0) = 0, \quad x(T) = x_T$$

   over all functions $x : [0,\pi] \to \mathbb{R}$.

   (a) Determine the Euler equation for this problem

   (b) Solve the Euler equation with $x(0) = 0, x(T) = x_T$

   (c) Are the second order conditions of Legendre satisfied?

3. Consider the system

$$\dot{x} = x(1-u), \qquad x(0) = 1, \quad x(1) = 4\,\mathrm{e}$$

   with cost

$$J = \int_0^1 -\ln(x(t)u(t))\,\mathrm{d}t.$$

   Since $x(0) > 0$ we have that $x(t) \geq 0$ for all t. For a well-defined cost we hence need $u(t) \in [0,\infty)$ but for the moment we allow any $u(t) \in \mathbb{R}$ and later verify that the optimal u_* is in fact > 0.

   (a) Determine the Hamiltonian

   (b) Determine the Hamiltonian equations

   (c) Show that $u = -1/(px)$ is the candidate optimal control

   (d) Substitute this u into the Hamiltonian equations and solve for $p_*(t)$ and then $x_*(t)$ and subsequently $u_*(t)$

   (e) Is $u_*(t) > 0$ for all $t \in [0,1]$?

4. Formulate the *principle of optimality* as used in Dynamic Programming

5. Suppose

$$\dot{x} = x + u, \qquad x(0) = 1$$

and that

$$J = 2x^2(T) + \int_0^T u^2(t)\,\mathrm{d}t$$

for arbitrary positive T.

(a) Determine the Riccati differential equation

(b) Solve the Riccati differential equation

(c) Determine the optimal state x_* and input u_* explicitly as functions of time

(d) Verify that $J(1, u_*) = P(0)$

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problem:	1	2	3	4	5
points:	2+2+1+1	2+3+1	1+2+2+3	3	1+2+2+3

Exam grade is $1 + 9p/p_{\max}$.

Euler:

$$\left(\frac{\partial}{\partial x} - \frac{d}{dt}\frac{\partial}{\partial \dot{x}}\right) F(t, x(t), \dot{x}(t)) = 0$$

Beltrami:

$$F - \frac{\partial F}{\partial \dot{x}}\dot{x} = C$$

Standard Hamiltonian equations for initial conditioned state:

$$\dot{x} = \frac{\partial H}{\partial p}(x, p, u), \qquad x(0) = x_0,$$

$$\dot{p} = -\frac{\partial H}{\partial x}(x, p, u), \qquad p(T) = \frac{\partial S}{\partial x}(x(T))$$

LQ Riccati differential equation:

$$\dot{P}(t) = -P(t)A - A^T P(t) + P(t)BR^{-1}B^T P(t) - Q, \qquad P(T) = G$$

Bellman:

$$\frac{\partial W(x,t)}{\partial t} + \min_{v \in \mathscr{U}} \left[\frac{\partial W(x,t)}{\partial x^T} f(x,v) + L(x,v)\right] = 0, \qquad W(x,T) = S(x)$$