Theory of Partial Differential Equations Name: ______________

191550105 | Final Exam Student ID #: ______________

2015.01.28 | 08:45–11.45 Major (opleiding): ______________

Exercise	01 (a)	(b)	02	03	04 **or** 05	Σ
Max	4	5	8	9	10	36
Grade						

Guidelines

- This is an open notes exam: you **may** consult your class notes, study notes, practice session notes and homework assignments.
- Work on **one** problem among 04 and 05.
- Theorems and formulas in homework or exercise problems **may be used without proof, unless stated otherwise.** Formulas for Fourier coefficients are always assumed known.
- If in doubt about anything, please **ask** the proctor for a clarification.

01. Consider the first order PDE

$$U_x + \pi U_y = U^3, \quad \text{with} \quad -\infty < x < \infty \quad \text{and} \quad y > 0. \tag{1}$$

(a) Determine the characteristics of (1).

(b) Equip (1) with the boundary data $U(x,0) = f(x)$; here, f is a given function defined for all real x. Find the solution $U(x,y)$.

02. Solve for U the following initial–boundary value problem for the diffusion equation:

$$\begin{aligned} U_t(x,t) &= U_{xx}(x,t), && \text{for all } 0 < x < 1 \text{ and } t > 0, \\ U(x,0) &= 1 + \cos(\pi x), && \text{for all } 0 \le x \le 1, \\ U_x(0,t) &= 0 \text{ and } U_x(1,t) = 0, && \text{for all } t > 0. \end{aligned} \tag{2}$$

[*Note: Derive the eigenvalues and eigenfunctions explicitly; do not just copy them from your notes. You may assume that all eigenvalues are non-positive, though.*]

03. Consider the Laplace equation posed outside the unit disk,

$$\begin{aligned} U_{xx} + U_{yy} &= 0, && \text{with } (x,y) \in D, \\ U(x,y) &= f(x,y), && \text{with } (x,y) \in \partial D, \end{aligned} \quad \text{and where} \quad \begin{aligned} D &= \left\{(x,y) \,|\, x^2 + y^2 > 1\right\}, \\ \partial D &= \left\{(x,y) \,|\, x^2 + y^2 = 1\right\}. \end{aligned} \tag{3}$$

Introduce inverted polar coordinates (R, Θ) through

$$(R,\Theta) = \left(\frac{1}{r}, \theta\right), \quad \text{where } (r,\theta) = \left(\sqrt{x^2+y^2}, \arctan\left(\frac{y}{x}\right)\right) \text{ are the ordinary polar coordinates.}$$

Write

$$V(R,\Theta) = U\big(x(R,\Theta), y(R,\Theta)\big) = U(x,y)$$

for the solution expressed in the new coordinates, and derive a boundary value problem for V. Then, solve that problem using known formulas. [Given: $\dfrac{\mathrm{d}\arctan(z)}{\mathrm{d}z} = \dfrac{1}{1+z^2}$.]

— SELECT AND SOLVE **ONLY ONE** OF THE FOLLOWING TWO PROBLEMS —

04. Consider, again, the initial-value problem posed in exercise 01:

$$\begin{aligned} U_x + \pi U_y &= U^3, \\ U(x,0) &= f(x), \end{aligned} \qquad \text{with} \quad -\infty < x < \infty \quad \text{and} \quad y > 0. \tag{4}$$

(a) The solution you derived may not exist over the entire half-plane

$$\mathbf{R} \times \mathbf{R}_+ = \{(x,y) \mid -\infty < x < \infty \quad \text{and} \quad y > 0\}. \tag{5}$$

Is this an artifact of the method of characteristics or a genuine problem with the PDE and/or the boundary condition? Also, discuss carefully the subset where your solution exists.

(b) Are there any solutions existing over the entire half-plane defined in (5)? If there are, find them all. If not, explain in detail why not.

05. Consider the wave equation posed in an ever-expanding interval,

$$\begin{aligned} U_{tt}(x,t) &= U_{xx}(x,t), && \text{with } (x,t) \in D, \\ U(-t,t) &= g(t), && \text{with } t > 0, \\ U(t,t) &= h(t), && \text{with } t > 0, \end{aligned} \qquad \text{and where } D = \{(x,t) \mid t > 0 \ \text{ and } \ \overset{-t}{\cancel{0}} < x < t\}. \tag{6}$$

Assume that $g(0) = h(0) = 0$, so that the initial displacement at the origin is zero, and find the solution $U(x,t)$.

Good luck!