**Exam for the course 201700034 "Introduction to PDEs"**
April 15, 2020, 8:45–11:45.

- **This online exam is held based on the assumption that you will do it yourselves, without the help from other people.**

  *By testing you remotely in this fashion, we express our trust that you will adhere to the ethical standard of behavior expected of you. This means that we trust you to answer the questions and perform the assignments in this test to the best of your own ability, without seeking or accepting the help of any source that is not explicitly allowed by the conditions of this test.*

- **In case we observe significant irregularities this will be reported to the Exam Committee.**

- **All answers must be motivated and clearly formulated.Try to draw and describe the physical system that is being modeled. Say in words what you are doing at each of the steps and do all of the calculations. Your solutions should make very clear to me that you understand ALL of the steps and the logic behind the steps. For all the new PDE material say WHY you are doing that step or making that choice.**

  During the exam you are only allowed to use the book: Applied Partial Differential Equations: with Fourier series and boundary value problems, Haberman R., Pearson.

- **Please write on the first page of your answer sheet:**

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  | *I made this test to the best of my own ability, without seeking or accepting the help of any source not explicitly allowed by the conditions of this test.* |

  **Sign this statement, and write below your signature your name, student number and study program.**

- Please indicate clearly all the questions you solve (i.e., 2(a), 2(b), . . . ) and avoid writing your solution at different, disjoint pages.

- Upload a scan of your answers, your student card, and if applicable your card for extra time, **as a pdf file** (preferably one file) in the Assignment Section of Canvas. Include your name in the file name.

  **Other formats than pdf will not be accepted.**

- After the end of the exam, or any earlier, you will have 15 minutes for the uploading of your answers into Canvas. After this, the site will be closed and no answers will be accepted. Hence uploading is possible till 12.00 for regular students and 12.45 for students with permission for extra time.

1. Consider initial-boundary value problem for the heat equation

$$u_t = k u_{xx}, \quad 0 \leqslant x \leqslant L, \quad t > 0,$$
$$u(0,t) = 1, \quad u_x(L,t) = 0,$$
$$u(x,0) = \cos\left(\frac{\pi}{2}x\right).$$

2pt (a) Show that the separation of variables method fails when the boundary conditions are nonhomogeneous. Discuss why it fails.

4pt (b) Solve the above-mentioned PDE. [Note: Derive the eigenvalues $\lambda_n$ and eigenfunctions $\Phi_n$, explicitly; do not just copy them from the book. ]

6pt 2. Consider the Laplace equation inside the semicircle $r \in [0,a]$ and $\theta \in [0,\pi]$, where the base of the semicircle is kept at zero temperature and the top arc is at a prescribed temperature, which is nonzero. Write down the PDE that describes this system. What is the additional condition that we require for $u(r,\theta)$ when $r = 0$? Using the separation of variables method, solve this problem.

3. Assume $u(x,t) = F(x+t) + G(x-t)$, solve the problem

$$u_{tt} - u_{xx} = 0, \quad 0 < x < 1, \quad t > 0$$
$$u(x,0) = 0, \quad u_t(x,0) = \sin(\pi x),$$
$$u(0,t) = 0, \quad u(1,t) = 0.$$

3pt (a) Use the change of variables $\xi = x+t$, $\eta = x-t$, to show that $u_{tt} - u_{xx} = -4\tilde{u}_{\xi\eta}$, where $\tilde{u}$ is the twice differentiable continuous function $u(x,t)$ expressed in the $\xi$, $\eta$ variables.

3pt (b) Use (a) to conclude that $u(x,t) = F(x+c)+G(x-t)$is the general solution to $u_{tt}-u_{xx} = 0$ and to derive the D'Alembert formula for the initial value problem of the wave equation in $-\infty < x < \infty$, with $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$ in .

4. Consider the following initial value problem for the continuously differentiable function $u(x,t)$

$$u_t + (1-2u)u_x = 0, \quad -\infty < x < \infty, \quad t > 0,$$
$$u(x,0) = \begin{cases} 1/4, & x < 0 \\ 1/2, & 0 < x < 1 \\ 1, & x > 1 \end{cases}$$

3pt (a) Give a physical interpretation of this problem.

3pt (b) Determine $u(x,t)$, and sketch the solution at various times.

5. For unknown $u(x,y)$, consider boundary value problem in the third quadrant ($x \leqslant 0$ and $y \leqslant 0$)

$$\nabla^2 u = f(x,y), \quad u(0,y) = g(y), \quad u(x,0) = h(x).$$

3pt (a) Using the method of images, find the Green function $G(x,y)$ for this problem. Hint: choose $f$ be the $\delta$ function and consider homogeneous boundary conditions.

3pt (b) Assuming that the Green function is known, provide the solution to the problem in terms of the Green function.

6pt

6. Solve the following PDE using the method of your choice. First discuss what physical system is being represented and what you expect to happen in the long term solution. Then choose a solution method and state why you picked the method that you picked and why the other methods are less suitable. After solving, does your solution match your expectation?

$$
\begin{aligned}
&u_t = u_{xx} + 1, \quad 0 \leqslant x \leqslant 1, \quad t > 0, \\
&u(0,t) = 0, \quad u(1,t) = 0, \\
&u(x,0) = 0.
\end{aligned}
$$

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The grade is determined by summing up all the points earned, dividing the sum by four and adding one.