Exam for Introduction to PDEs
April 14, 2021, 09:00–11:00.

- **This closed-book online exam is held based on the assumption that you will do it yourselves, without the help from other people and the lecture materials.**

  By testing you remotely in this fashion, we express our trust that you will adhere to the ethical standard of behavior expected of you. This means that we trust you to answer the questions and perform the assignments in this test to the best of your own ability, without seeking or accepting the help of any source that is not explicitly allowed by the conditions of this test.

- **In case we observe significant irregularities this will be reported to the Exam Committee. You must complete the test at home. At the end of the test you take pictures of your exam and create a single PDF file containing your test, then you upload the test PDF onto Canvas. You can also write the test on a tablet. Include your name on the first page and in the filename of your document. Also include a picture of your student card (and your extra time card if relevant). The results of this exam are preliminary and will only become permanent when we are convinced the vast majority of you deserve to obtain this grade. We reserve the right to perform some short oral exams on a selection of students to check whether your understanding of the material is in line with the work which was handed in.**

- **Please read the following paragraph carefully, and copy the text below it verbatim to your answer sheet:**

  I made this test to the best of my own ability, without seeking or accepting the help of any source not explicitly allowed by the conditions of this test.

  PLACE YOUR STUDENT CARD (AND EXTRA TIME CARD) ON YOUR DESK NOW, READY TO BE INCLUDED WITH YOUR ANSWERS.

- **All answers must be motivated and clearly formulated. Please indicate clearly all the questions you solve (i.e., 2(a), 2(b), . . .) and avoid writing your solution at different, disjoint pages.**

- **The Fourier transform table is added to the last page of this exam.**

1. Consider

$$\rho(x)u_{tt} = T_0 u_{xx} + \alpha(x)u + \beta(x)u_t, \quad 0 < x < L, \quad t > 0$$
$$u(x,0) = f(x), \quad u_t(x,0) = g(x),$$
$$u(0,t) = 0, \quad u(L,t) = 0.$$

where $\rho(x) > 0$, $\alpha(x) < 0$, $\beta(x) < 0$, for all x in $[0, L]$, and T_0 is a positive constant.

2pt (a) Give a physical interpretation of this problem.

3pt (b) Show that separation of variables works only if $\beta(x) = c\rho(x)$, where c is a constant.

2pt (c) Assume that $\beta(x) = 0$, show that the eigenvalues are positive if $\alpha(x) < 0$. (Hint: Use separation of variables to obtain the spatial ODE $\Phi(x)$ and the time-dependent ODE $H(t)$, then solve the eigenvalue problem for Φ in the Sturm-Liouville form for the spatial component.)

5pt (d) Assume that $\rho(x) = T_0 = 1$, $\alpha(x) = g(x) = 0$, $\beta(x) = -b$, where $0 < b < 1$, and $L = 1$. Solve the given initial-boundary value problem by separation of the variables.

2. Consider the following boundary value problem in the wedge $\Omega = \{(x,y) \in \mathbb{R}^2,\ y > 0,\ x > y\}$,

$$\nabla^2 u = f(x,y), \quad u(x,x) = g(x), \quad u(x,0) = h(x).$$

2pt (a) Give a physical interpretation of the problem.

4pt (b) Using the method of images, find the Green's function $G(x,y)$ for this problem. Hint: choose f be the δ function and consider homogeneous boundary conditions.

6pt (c) Assuming that the Green function is known, provide the solution to the problem in terms of the Green function.

4pt 3. (a) Suppose that we want $h(x)$ but know its Fourier sine transform $H(\omega)$ to be a product

$$H(\omega) = S(\omega)C(\omega),$$

where $S(\omega)$ is the sine transform of $s(x)$ and $C(\omega)$ is the cosine transform of $c(x)$. Assuming that $c(x)$ is even and $s(x)$ is odd, show that

$$h(x) = \frac{1}{\pi}\int_0^\infty s(\bar{x})[c(x-\bar{x}) - c(x+\bar{x})]\, d\bar{x} = \frac{1}{\pi}\int_0^\infty c(\bar{x})[s(x+\bar{x}) - s(\bar{x}-x)]\, d\bar{x}$$

You can show one of the equalities.

2pt (b) Consider the following initial value problem and discuss what physical system is being represented.

$$u_t = ku_{xx}, \quad 0 < x < \infty, \quad t > 0,$$
$$u(0,t) = 1,\ u(x,0) = f(x).$$

6pt (c) Solve the given problem in 3(b) by applying either the Fourier sine or cosine transform method. Motivate your choice for the solution method.

The grade is determined by summing up all the points earned, dividing the sum by four and adding one.

Table 10.5.2: Fourier Cosine Transform

$f(x) = \int_0^\infty F(\omega)\cos\omega x \, d\omega$	$C[f(x)] = F(\omega)$ $= \dfrac{2}{\pi}\displaystyle\int_0^\infty f(x)\cos\omega x \, dx$	Reference
$\dfrac{df}{dx}$	$-\dfrac{2}{\pi}f(0) + \omega S[f(x)]$	Derivatives (Sec. 10.5.4)
$\dfrac{d^2 f}{dx^2}$	$-\dfrac{2}{\pi}\dfrac{df}{dx}(0) - \omega^2 F(\omega)$	Derivatives (Sec. 10.5.4)
$\dfrac{\beta}{x^2+\beta^2}$	$e^{-\omega\beta}$	Exercise 10.5.1
$e^{-\epsilon x}$	$\dfrac{2}{\pi}\cdot\dfrac{\epsilon}{\epsilon^2+\omega^2}$	Exercise 10.5.2
$e^{-\alpha x^2}$	$2\dfrac{1}{\sqrt{4\pi\alpha}}e^{-\omega^2/4\alpha}$	Exercise 10.5.3
$\int_0^\infty g(\overline{x})[f(x-\overline{x}) + f(x+\overline{x})]d\overline{x}$	$F(\omega)G(\omega)$	Convolution (Exercise 10.5.7)

Table 10.5.1: Fourier Sine Transform

$f(x) = \int_0^\infty F(\omega)\sin\omega x \, d\omega$	$S[f(x)] = F(\omega)$ $= \dfrac{2}{\pi}\displaystyle\int_0^\infty f(x)\sin\omega x \, dx$	Reference
$\dfrac{df}{dx}$	$-\omega C[f(x)]$	Derivatives (Sec. 10.5.4)
$\dfrac{d^2 f}{dx^2}$	$\dfrac{2}{\pi}\omega f(0) - \omega^2 F(\omega)$	Derivatives (Sec. 10.5.4)
$\dfrac{x}{x^2+\beta^2}$	$e^{-\omega\beta}$	Exercise 10.5.1
$e^{-\epsilon x}$	$\dfrac{2}{\pi}\cdot\dfrac{\omega}{\epsilon^2+\omega^2}$	Exercise 10.5.2
1	$\dfrac{2}{\pi}\cdot\dfrac{1}{\omega}$	Exercise 10.5.9
$\dfrac{1}{\pi}\displaystyle\int_0^\infty f(\overline{x})[g(x-\overline{x}) - g(x+\overline{x})]d\overline{x}$ $= \dfrac{1}{\pi}\displaystyle\int_0^\infty g(\overline{x})[f(x+\overline{x}) - f(\overline{x}-x)]d\overline{x}$	$S[f(x)]C[g(x)]$	Convolution (Exercise 10.5.6)

$f(x) = \int_{-\infty}^{\infty} F(\omega) e^{-i\omega x}\, d\omega$	$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x) e^{i\omega x}\, dx$	Reference
$e^{-\alpha x^2}$	$\frac{1}{\sqrt{4\pi\alpha}} e^{-\omega^2/4\alpha}$	Gaussian (Sec. 10.3.3)
$\sqrt{\frac{\pi}{\beta}} e^{-x^2/4\beta}$	$e^{-\beta\omega^2}$	Gaussian (Sec. 10.3.3)
$\frac{\partial f}{\partial t}$	$\frac{\partial F}{\partial t}$	Derivatives (Sec. 10.4.2)
$\frac{\partial f}{\partial x}$	$-i\omega F(\omega)$	Derivatives (Sec. 10.4.2)
$\frac{\partial^2 f}{\partial x^2}$	$(-i\omega)^2 F(\omega)$	Derivatives (Sec. 10.4.2)
$\frac{1}{2\pi}\int_{-\infty}^{\infty} f(\overline{x}) g(x-\overline{x}) d\overline{x}$	$F(\omega)G(\omega)$	Convolution (Sec. 10.4.3)
$\delta(x - x_0)$	$\frac{1}{2\pi} e^{i\omega x_0}$	Dirac delta function (Exercise 10.3.18)
$f(x-\beta)$	$e^{i\omega\beta} F(\omega)$	Shifting theorem (Exercise 10.3.5)
$xf(x)$	$-i\frac{dF}{d\omega}$	Multiplication by x (Exercise 10.3.8)
$\frac{2\alpha}{x^2+\alpha^2}$	$e^{-\lvert\omega\rvert\alpha}$	Exercise 10.3.7
$f(x) = \begin{cases} 0 & \lvert x\rvert > a \\ 1 & \lvert x\rvert < a \end{cases}$	$\frac{1}{\pi}\frac{\sin a\omega}{\omega}$	Exercise 10.3.6

Table 10.4.1: Fourier Transform

Rayleigh quotient:

$$\lambda = \frac{-p\phi\, d\phi/dx\big|_a^b + \int_a^b [p(d\phi/dx)^2 - q\phi^2]\, dx}{\int_a^b \phi^2 \sigma\, dx},$$

Product-to-sum[30]
$2\cos\theta\cos\varphi = \cos(\theta-\varphi) + \cos(\theta+\varphi)$
$2\sin\theta\sin\varphi = \cos(\theta-\varphi) - \cos(\theta+\varphi)$
$2\sin\theta\cos\varphi = \sin(\theta+\varphi) + \sin(\theta-\varphi)$
$2\cos\theta\sin\varphi = \sin(\theta+\varphi) - \sin(\theta-\varphi)$