Exam for the course 201700034 "Introduction to PDEs"
April 20, 2022, 8:45–11:45.

The use of any electronic devices and lecture material is not allowed. All answers must be motivated and clearly formulated. Please indicate clearly all the questions you solve (i.e., 2(a), 2(b), ...) and avoid writing your solution at different, disjoint pages.

1. Consider

$$\begin{aligned} &u_{tt} = u_{xx}, \quad x > 0, \quad t > 0, \\ &u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad x > 0, \\ &a u_x(0,t) = b u(0,t) - h(t), \quad t > 0. \end{aligned}$$

where a and b are constants.

1pt (a) Give a physical interpretation of this problem.

1pt (b) Assume that $a = 1, b = 0$. Under which condition is the method of separation of variables generally applicable?

5pt (c) Assume that $a = 0$, $b = 1$, and $f \equiv 0$. The general solution of the PDE is given by

$$u(x,t) = F(x-t) + G(x+t)$$

Determine the solution of the initial-boundary value problem by using the general solution of the PDE.

3pt (d) Assume that $a = 1$, $b = 0$ and $h(t) = 0$. Solve the given initial-boundary value problem by using the method of your choice.

2. Consider the following initial value problem for the continuously differentiable function $u(x,t)$

$$\begin{aligned} &u_t + A u_x = B, \quad -\infty < x < \infty, \quad t > 0, \\ &u(x,0) = f(x). \end{aligned}$$

3pt (a) Assume that $A = t$, $B = 5$ with the given the initial condition $u(x,0) = f(x)$. Determine an explicit solution of $u(x,t)$ by using the system of characteristic equations.

1pt (b) Assume that $A = (1-2u)$, $B = 0$, and $f(x) = \begin{cases} 1/3, & x < 0, \\ 1/2, & 0 < x < 1, \\ 2/3, & x > 1. \end{cases}$.

Give a physical interpretation of this problem.

6pt (c) Determine the solution $u(x,t)$ of the given problem in 2(b) and sketch the solution at various times.

3. Consider the following initial- boundary value problem for the heat equation

$$\begin{aligned} &u_t = k u_{xx}, \quad 0 \leqslant x \leqslant L, \quad t > 0, \\ &u(0,t) = 0, \quad u_x(L,t) = 1, \\ &u(x,0) = \cos(\pi x). \end{aligned}$$

2pt (a) Determine the steady-state temperature distribution $u_E(x)$.

4pt (b) Find the initial boundary value problem satisfied by the transient temperature distribution $v(x,t)$, and solve this problem. (Hint: $u(x,t) = u_E(x) + v(x,t)$. The solution $v(x,t)$ must be expressed in function of the eigenvalues λ_n of the corresponding Sturm-Liouville problem. Assume that all the eigenvalues λ_n are positive.)

See the other side

3pt 4. (a) Show that the Fourier sine transform of $e^{-\alpha x}$, where $\alpha > 0$ and $x \geq 0$, is $\dfrac{2}{\pi}\dfrac{\omega}{\alpha^2+\omega^2}$ (derive the relation step by step).

1pt (b) Consider the following initial value problem and discuss what physical system is being represented.

$$u_{xx} + u_{yy} = 0, \quad 0 < x < \infty, \quad 0 < y < \infty,$$
$$u(0, y) = g(y), \;\; u_y(x, 0) = 0.$$

6pt (c) Solve the given problem in 3(b) by applying either the Fourier sine or cosine transform method. Motivate your choice for the solution method.

The grade is determined by summing up all the points earned, dividing the sum by four and adding one.

Table 10.5.2: Fourier Cosine Transform

$f(x) = \int_0^\infty F(\omega)\cos\omega x\, d\omega$	$C[f(x)] = F(\omega) = \frac{2}{\pi}\int_0^\infty f(x)\cos\omega x\, dx$	Reference
$\frac{df}{dx}$	$-\frac{2}{\pi}f(0) + \omega S[f(x)]$	Derivatives (Sec. 10.5.4)
$\frac{d^2 f}{dx^2}$	$-\frac{2}{\pi}\frac{df}{dx}(0) - \omega^2 F(\omega)$	
$\frac{\beta}{x^2+\beta^2}$	$e^{-\omega\beta}$	Exercise 10.5.1
$e^{-\epsilon x}$	$\frac{2}{\pi}\cdot\frac{\epsilon}{\epsilon^2+\omega^2}$	Exercise 10.5.2
$e^{-\alpha x^2}$	$2\frac{1}{\sqrt{4\pi\alpha}}e^{-\omega^2/4\alpha}$	Exercise 10.5.3
$\int_0^\infty g(\bar{x})[f(x-\bar{x}) + f(x+\bar{x})]d\bar{x}$	$F(\omega)G(\omega)$	Convolution (Exercise 10.5.7)

Table 10.5.1: Fourier Sine Transform

$f(x) = \int_0^\infty F(\omega)\sin\omega x\, d\omega$	$S[f(x)] = F(\omega) = \frac{2}{\pi}\int_0^\infty f(x)\sin\omega x\, dx$	Reference
$\frac{df}{dx}$	$-\omega C[f(x)]$	Derivatives (Sec. 10.5.4)
$\frac{d^2 f}{dx^2}$	$\frac{2}{\pi}\omega f(0) - \omega^2 F(\omega)$	
$\frac{x}{x^2+\beta^2}$	$e^{-\omega\beta}$	Exercise 10.5.1
$e^{-\epsilon x}$	$\frac{2}{\pi}\cdot\frac{\omega}{\epsilon^2+\omega^2}$	Exercise 10.5.2
1	$\frac{2}{\pi}\cdot\frac{1}{\omega}$	Exercise 10.5.9
$\frac{1}{\pi}\int_0^\infty f(\bar{x})[g(x-\bar{x}) - g(x+\bar{x})]d\bar{x} = \frac{1}{\pi}\int_0^\infty g(\bar{x})[f(x+\bar{x}) - f(\bar{x}-x)]d\bar{x}$	$S[f(x)]C[g(x)]$	Convolution (Exercise 10.5.6)

$f(x) = \int_{-\infty}^{\infty} F(\omega) e^{-i\omega x}\, d\omega$	$F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{i\omega x}\, dx$	Reference
$e^{-\alpha x^2}$	$\frac{1}{\sqrt{4\pi\alpha}} e^{-\omega^2/4\alpha}$	Gaussian (Sec. 10.3.3)
$\sqrt{\frac{\pi}{\beta}} e^{-x^2/4\beta}$	$e^{-\beta\omega^2}$	Gaussian (Sec. 10.3.3)
$\frac{\partial f}{\partial t}$	$\frac{\partial F}{\partial t}$	Derivatives (Sec. 10.4.2)
$\frac{\partial f}{\partial x}$	$-i\omega F(\omega)$	Derivatives (Sec. 10.4.2)
$\frac{\partial^2 f}{\partial x^2}$	$(-i\omega)^2 F(\omega)$	Derivatives (Sec. 10.4.2)
$\frac{1}{2\pi} \int_{-\infty}^{\infty} f(\overline{x}) g(x - \overline{x}) d\overline{x}$	$F(\omega)G(\omega)$	Convolution (Sec. 10.4.3)
$\delta(x - x_0)$	$\frac{1}{2\pi} e^{i\omega x_0}$	Dirac delta function (Exercise 10.3.18)
$f(x - \beta)$	$e^{i\omega\beta} F(\omega)$	Shifting theorem (Exercise 10.3.5)
$x f(x)$	$-i \frac{dF}{d\omega}$	Multiplication by x (Exercise 10.3.8)
$\frac{2\alpha}{x^2 + \alpha^2}$	$e^{-\lvert\omega\rvert\alpha}$	Exercise 10.3.7
$f(x) = \begin{cases} 0 & \lvert x\rvert > a \\ 1 & \lvert x\rvert < a \end{cases}$	$\frac{1}{\pi} \frac{\sin a\omega}{\omega}$	Exercise 10.3.6

Table 10.4.1: Fourier Transform

Rayleigh quotient:

$$\lambda = \frac{-p\phi\, d\phi/dx\big|_a^b + \int_a^b \left[p(d\phi/dx)^2 - q\phi^2\right] dx}{\int_a^b \phi^2 \sigma\, dx},$$

Product-to-sum[30]
$2\cos\theta\cos\varphi = \cos(\theta - \varphi) + \cos(\theta + \varphi)$
$2\sin\theta\sin\varphi = \cos(\theta - \varphi) - \cos(\theta + \varphi)$
$2\sin\theta\cos\varphi = \sin(\theta + \varphi) + \sin(\theta - \varphi)$
$2\cos\theta\sin\varphi = \sin(\theta + \varphi) - \sin(\theta - \varphi)$