EXAM Partial Differential Equations (155010)

Wednesday, January 30, 2008, 09:00 - 12:00

Maybe a part of scores to **2**.b, **4**.c and **5**.c has already been earned by homeworks. This score is marked by "_" in the grading at the end of this exam.

1. Consider the linear first order PDE problem for $u = u(x, y)$

$$2u_x + 3u_y + 8u = 0$$

a) Give the characteristic equation and determine the characteristics.

b) Find a solution $u(x, y)$ of the PDE having values $u(x, x) = x^4$.

2.

a) Prove that the equation

$$x^2 u_{xx} - 2xyu_{xy} + y^2 u_{yy} + xu_x + yu_y = 0$$

is parabolic and find its canonical form.

b) Furthermore, find the general solution on the half plane $x > 0$.

3. Let $u(x, t)$ be a solution of the wave equation

$$u_{tt} - 9u_{xx} = 0 \quad (-\infty < x < \infty,\ t > 0)$$

with Cauchy data

$$u(x, 0) = u_t(x, 0) = \begin{cases} 1, & \text{if } |x| \leq 2; \\ 0, & \text{if } |x| > 2. \end{cases}$$

a) Use d'Alambert solution formula to calculate $u(0, \frac{2}{3})$.

b) Explain why $u(0, \frac{2}{3})$ is the maximum value of $u(x, t)$.

4. Assume that the function u is a solution of the diffusion equation $u_t = ku_{xx}$ for $0 < x < L,\ t > 0$.

a) Show that u satisfies the integral expression

$$\frac{d}{dt}\int_0^L \frac{1}{2}u^2(x,t)\,dx = ku(x,t)u_x(x,t)\Big|_{x=0}^{x=L} - k\int_0^L u_x^2(x,t)\,dx.$$

b) Show that the initial-boundary-value problem for the heat equation

$$u_t = ku_{xx},\quad u(0,t) = u(L,t) = 0,\quad u(x,0) = f(x),\quad \text{for } 0 \le x \le L,\quad t \ge 0,$$

with f a given function, can have only one solution.
(Hint: make use of a)).

c) Give the formal general solution as found by separation of variables for the problem in b).

5.

a) Derive the Laplace equation

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$$

for polar coordinates (r,θ) from its rectangular coordinates form

$$u_{xx} + u_{yy} = 0.$$

b) Solve the Dirichlet problem $\nabla^2 u = 0$ on the unit disk $x^2 + y^2 \le 1$ subject to the boundary condition $u(x,y) = y^2$ for $x^2 + y^2 = 1$.

c) Consider the Dirichlet problem for the disk $r < \rho$ with $u(\rho,\theta) = f(\theta)$. Suppose f is periodic of period 2π and is an odd function of θ on $[-\pi,\pi]$. Show that the solution $u(r,\theta)$ is also an odd function of θ. If f is an even function of θ, is the solution even in θ?

Grading

1 a) 2	**2 a)** 4	**3 a)** 3	**4 a)** 3	**5 a)** 2
b) 4	**b)** 3	**b)** 3	**b)** 3	**b)** 3
			c) 3	**c)** 3

Total: $36 + 4 = 40$ points.