UNIVERSITEIT
TWENTE.

Applied Analysis Group
Department of Applied Mathematics
Christoph Brune

Course: Applied Functional Analysis (191506302)

Date: Tuesday, Jan 31, 2017
Time: 8:45-11:45

- An explanation to every answer is required.
- You can make use of a calculator.
- (Part of) the scores to exercises **4**(c),(d) and **6** may be earned by homeworks, see the underlined numbers in the table at the end of the exam.

---

1. The linear space $C[0,1]$ of continuous real functions on the interval $[0,1]$ is equipped with

$$||f||_\infty = \max_{0\le t\le 1} |f(t)|$$

and, as an alternative, with

$$||f|| = |f(0)| + \int_0^1 |f(t)|\, dt \,.$$

(a) Show that $||f||$ defines a norm on $C[0,1]$.

(b) Prove that the norms $||\cdot||_\infty$ and $||\cdot||$ are not equivalent on $C[0,1]$.

(c) Give an example of a linear functional $A : C[0,1] \to \mathbb{R}$ which is continuous with respect to $||\cdot||_\infty$, but not with respect to $||\cdot||$.

(d) From the course it is known that $C[0,1]$ equipped with $||\cdot||_\infty$ is a Banach space. Is $C[0,1]$ also complete with respect to $||\cdot||$?

2. Let an operator $A : \ell^2(\mathbb{C}) \to \ell^2(\mathbb{C})$ be defined as

$$A(a_1, a_2, a_3, \cdots) := (ia_2, -ia_1, \frac{1}{2}ia_4, -\frac{1}{2}ia_3, \frac{1}{3}ia_6, \cdots)\,,$$

that is $(A\underline{a})_{2n-1} := \frac{1}{n}ia_{2n}$ and $(A\underline{a})_{2n} := -\frac{1}{n}ia_{2n-1}$.

(a) Prove that A is self-adjoint.

(b) Prove that A is compact.

(c) Determine the spectrum $\sigma(A)$.

3. Consider the real Hilbert space $\ell^2(\mathbb{R})$ with the usual inner product. Let the set $T = \{y_1, y_2, \cdots\}$ consist of vectors $y_1, y_2, \cdots$ in $\ell^2(\mathbb{R})$ of the following form

$$y_1 = (1,0,-2,0,0,\cdots),\quad y_2 = (0,1,0,-2,0,0,\cdots),\quad y_3 = (0,0,1,0,-2,0,0,\cdots),\quad \text{etc.}$$

(a) Find an orthonormal basis (that is a maximal orthonormal system) for $T^\perp$.

(b) Let $z = (1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots)$. Find the vector $h \in T^\perp$ which satisfies

$$||z - h|| = \min_{x\in T^\perp} ||z - x||\,.$$

4. Consider the space $L^2(0,1)$, let $F := \{c\mathbb{1} \mid c \in \mathbb{C}\}$ be the space of constant functions, and let $Q : L^2(0,1) \to F$ be the orthogonal projection onto F. Furthermore, let $J : L^2(0,1) \to L^2(0,1)$ be the integration operator, i.e. $Jf(x) = \int_0^x f(t)dt$.

$A(f)\ J(Q(f) - J(f))$

(a) Show that the operator $A := J(Q - J)$ is an integral operator with a kernel function. Determine the kernel function and show that A is bounded.

(b) Show that $A^* = A$.

(c) Show that $||A|| \leq \frac{1}{\sqrt{6}}$.

(d) Show that if $f \in L^2(0,1)$, then $u = Af$ satisfies

$$u \in \mathrm{H}^2(0,1) := W^{2,2}(0,1), \quad u'' = -f, \quad u(0) = u'(1) = 0 \,.$$

5. Let H be a Hilbert space and $T : H \to H$ a bounded linear operator. A linear subspace $Y \subset H$ is said to be *invariant under* T if $T(Y) \subset Y$.

(a) Show that a closed linear subspace Y of H is invariant under T if and only if $Y^\perp$ is invariant under T^*.

(b) Give an example of a non-trivial closed linear subspace Y of $L^2(0,1)$ which is invariant under T where T is defined by $Tf(x) = \int_0^x f(t)dt$. (non-trivial means: $Y \neq \{0\}$ and $Y \neq L^2(0,1)$)

6. Let M be a linear subspace of a normed vector space E and N be a linear subspace of E^* (the dual space of E). We define

$$\begin{aligned} M^0 &= \{\varphi \in E^* \mid \varphi(x) = 0 \text{ for all } x \in M\} \\ N_0 &= \{x \in E \mid \varphi(x) = 0 \text{ for all } \varphi \in N\} \end{aligned}$$

(a) Take $E = L^3(-1,1)$ and $M = \{f \in L^3(-1,1) \mid f(x) = 0 \text{ for all } 0 \leq x \leq 1\}$. Identify M^0 as a *subspace* of $L^{\frac{3}{2}}(-1,1)$. $L^{\frac{3}{2}}(-1,1)$ is the dual space of E. *Which functions* are elements of M^0?

(b) Show (in general) that $(M^0)_0 = M$ holds in case M is closed. Hint: Use the theorem of Hahn-Banach.

Grading scheme:

1. (a) 1	**2.** (a) 2	**3.** (a) 3	**4.** (a) 3	**5.** (a) 4	**6.** (a) 3
(b) 1	(b) 2	(b) 2	(b) 1	(b) 3	(b) 3
(c) 1	(c) 2		(c) 1		
(d) 2			(d) 2		

Total: $36 + 4 = 40$ points