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Applied Functional Analysis (150630)

Friday, January 26, 2007 09.00–12.00

- an explanation to every answer is required.
- you can make use of a calculator.
- (part of) the score to exercises 1(c) and 4(a), (b), (c) may be earned by homeworks; see the score table below (these are indicated by $\underline{\cdot}$).

1. The normed vector space of all bounded real sequences is denoted by ℓ^∞, its norm by $|\underline{a}|_\infty = \sup_{n\in\mathbb{N}} |a_n|$, where $\underline{a} = (a_1, a_2, a_3, \dots)$ with $a_n \in \mathbb{R}$.

   (a) Check the triangle inequality for $|\ |_\infty$.

   (b) Give an example of two vectors $\underline{a}$, $\underline{b} \in \ell^\infty$ for which
   $|\underline{a}+\underline{b}|_\infty^2 + |\underline{a}-\underline{b}|_\infty^2 = 2\,|\underline{a}|_\infty^2 + 2\,|\underline{b}|_\infty^2$ does not hold.
   What conclusion can be taken for ℓ^∞

   We define $c_0 = \{\underline{a} \in \ell^\infty |\ \lim_{n\to\infty} a_n = 0\}$ to be the linear subspace of all sequences which tend to zero.

   (c) Show that c_0 is a closed subset of ℓ^∞.

2. Consider the boundary value problem for y:

$$\begin{cases} y' - y &= f \\ y(0) &= 0 \end{cases}$$

   where f is a given function in $L^2(0,1)$.

   (a) Show that $y(x) = \int_0^x e^{x-t} f(t)\,dt$ is a solution to the problem.
   Define the integral operator $A : L^2(0,1) \to L^2(0,1)$ by $Ag(x) = \int_0^1 k(x,t)g(t)\,dt$ for $g \in L^2(0,1)$ where

$$k(x,t) = \begin{cases} e^{x-t}, & \text{if } 0 \le t < x \le 1 \\ 0, & \text{if } 0 \le x \le t \le 1. \end{cases}$$

(b) Show that $y(x)$ from (a) equals $Af(x)$.

(c) Prove that A is bounded (give an upper bound for $\|A\|$).

3. We define $A : \ell_2 \to \ell_2$ to be $A(a_1, a_2, \dots) = (a_1 + a_2, a_2 + a_3, a_3 + a_4, \dots)$

   (a) Prove that A is not invertible.

   (b) Show that $\sum_{i=1}^{\infty} |a_i a_{i+1}| \leq \sum_{i=1}^{\infty} a_i^2$.

   Hint: Consider the inner product between the vectors $(|a_1|, |a_2|, \dots)$ and $(|a_2|, |a_3|, \dots)$.

   (c) Prove that A is bounded with $\|A\| = 2$.

4. Let E be an infinite dimensional complex Banach space. Suppose $A : E \to E$ is bounded and linear and has the property that $A^k = 0$ for some positive natural number k.

   (a) Prove that $\lambda = 0$ is an eigenvalue of A.

   (b) Show that $\sigma(A) = \{0\}$.

   Hint: Use the Neumann series.

   (c) Give an example of a non-zero operator $A : \ell_2 \to \ell_2$ for which $A^2 = 0$.

5. Let $A : L^2(0,1) \to L^2(0,1)$ be the operator on the complex Hilbert space $L^2(0,1)$, defined by

$$Af(x) = \int_0^x f(t)\, dt \qquad \text{for all } f \in L^2(0,1).$$

   (a) Show that A does not have any eigenvalue.

   (b) Determine A^*.

   (c) Prove that AA^* is an integral operator of the type

$$AA^*g(x) = \int_0^1 k(x,y)g(y)\, dy \qquad \text{for all } g \in L^2(0,1).$$

   (d) It has been proved that the eigenvalues of AA^* are

$$\lambda_n = \frac{1}{\pi^2} \frac{4}{(2n+1)^2} \qquad \text{with } n \in \mathbb{Z}.$$

   Determine the spectrum $\sigma(AA^*)$ of AA^*.

Grading Points

1.	(a)	2	**2.**	(a)	3	**3.**	(a)	2	**4.**	(a)	2	**5.**	(a)	2
	(b)	2		(b)	2		(b)	2		(b)	2		(b)	2
	(c)	3		(c)	2		(c)	3		(c)	2		(c)	3
													(d)	2

Total: $36 + 4 = 40$ points.