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Applied Functional Analysis (150630)

Friday, January 25, 2008, 09.00-12.00

- An explanation to every answer is required.
- You can make use of a calculator.
- (Part of) the score to exercise 1(d) and 3(c),(d) may be earned by homeworks; see the score table below (these are indicated by $\underline{.}$).

1. The linear space $C[0,1]$ of continuous real functions on the interval $[0,1]$ is endowed with
$$\|f\|_\infty = \max_{0\le t\le 1} |f(t)|$$
and
$$\|f\| = |f(0)| + \int_0^1 |f(t)|\,dt.$$
   (a) Show that $\|f\|$ defines a norm on $C[0,1]$.
   (b) Prove that both norms $\|\ \|_\infty$ and $\|\ \|$ are non-equivalent norms on $C[0,1]$.
   (c) Give an example of a linear functional $A : C[0,1] \to \mathbf{R}$ which is continuous with respect to $\|\ \|_\infty$, but not with respect to $\|\ \|$.
   (d) From the course it is known that $C[0,1]$ endowed with $\|\ \|_\infty$ is a Banach space. Is $C[0,1]$ also complete with respect to $\|\ \|$?

2. By $L^2(0,2\pi)$ we denote the real vector space of all (classes of) real square-integrable functions on $(0,2\pi)$, endowed with the usual inner product
$$(f,g) = \int_{x=0}^{2\pi} f(x)g(x)\,dx.$$
Let D be the closed linear subspace,
$$D = \left\{ f \in L^2(0,2\pi) \Big| \int_{x=0}^{2\pi} f(x)\,dx = 0 \right\}.$$

(a) Determine $D^{\perp}$, the orthoplement of D in $L^2(0,2\pi)$.

(b) Find the best approximation in D to $g(x) = x^2$.

During the course it was demonstrated that $\{g_0, g_1, h_1, g_2, h_2, \ldots\}$, (with $g_0(x) = \frac{1}{\sqrt{2\pi}}, g_k(x) = \frac{1}{\sqrt{\pi}}\cos(kx), h_k(x) = \frac{1}{\sqrt{\pi}}\sin(kx)$) is a maximal orthonormal system (MOS) for $L^2(0,2\pi)$

(c) Give a MOS for the subspace D.

3. The linear operator $A : l^2 \to l^2$ is given by

$$A(b) = (\beta_2, \beta_2, \beta_4, \beta_4, \beta_6, \beta_6, \ldots) \text{ if } b = (\beta_1, \beta_2, \beta_3, \ldots)\,.$$

(a) Show that A is bounded and determine $\|A\|$.

(b) Calculate A^2 and A^*.

(c) Is A a compact operator? Exlain your answer.

(d) Find the spectrum $\sigma(A)$ of A.

4. Let H be a Hilbert space and $T : H \to H$ a bounded linear operator. A linear subspace $Y \subset H$ is said to be **invariant under** T if $T(Y) \subset Y$.

(a) Show that a closed linear subspace Y of H is invariant under T if and only if $Y^{\perp}$ is invariant under T^*.

(b) Give an example of a non-trivial closed linear subspace Y of $L^2(0,1)$ which is invariant under T where T is defined by $Tf(x) = \int_0^x f(t)\,dt$. (Non-trivial means: $Y \neq \{0\}$ and $Y \neq L^2(0,1)$).

5. Consider $A : L^2(0,\infty) \to L^2(0,\infty)$ given by

$$Af(x) = \int_0^x f(y)dy$$

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(a) Prove that A is unbounded.

(b) Determine A^*.

GRADING POINTS

1.(a)2	2.(a)2	3.(a)2	4.(a)3	5.(a)2
(b)2	(b)2	(b)2	(b)3	(b)3
(c)2	(c)2	(c)2		
(d)3		(d)4		

Total: 36+4=40 points.